

Revisiting the Schwarzschild and the Hilbert-Droste Solutions of Einstein Equation and the Maximal Extension of the Latter*

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Abstract

In this pedagogical note, the differences between the Schwarzschild and the Hilbert-Droste solutions of Einstein equation are scrutinized through a rigorous mathematical approach, based on the idea of warped product of manifolds. It will be shown that those solutions are indeed *different* because the topologies of the manifolds corresponding to them are different. After establishing this fact beyond any doubt, the maximal extension of the Hilbert-Droste solution (the Kruskal-Szekeres spacetime) is derived with details and its topology compared with the ones of the Schwarzschild and the Hilbert-Droste solution.

We also study the problem of the imbedding of the Hilbert-Droste solution in a vector manifold, hopefully clarifying the work of Kasner and Fronsdal on the subject.

In an Appendix, we present a rigorous discussion of the Einstein-Rosen Bridge. A comprehensive bibliography of the historical papers involved in our work is given at the end.

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1 Introduction

The journal *General Relativity and Gravitation* reprinted in 2003 the famous paper in which Schwarzschild consecrated himself as the first person to find an exact solution of the Einstein field equation (cf. ref. [1]). Following the same volume of that journal, S. Antoci and D.-E. Liebscher published an editorial note claiming that the solution presented by Schwarzschild in 1916 (which describes the gravitational field generated by a point of mass) is not equivalent to the one currently taught in textbooks on General Relativity. The latter being a solution which was, however, found by J. Droste and D. Hilbert just a year after Schwarzschild's publication. This event culminated in a series of papers concerned with the equivalence or the nature of these two solutions.

Three years after the editorial note of Antoci and Liebscher, a rectification was published in the above journal (cf. ref. [2]) claiming that the solutions of

Schwarzschild and of Hilbert-Droste are indeed equivalent, based on the existence of a coordinate transformation for which the metric found originally by Schwarzschild can be written in the same coordinate form as the one found by Droste and Hilbert. This opinion is shared by the authors of ref. [3], published in 2007, and of ref. [4], published in 2013.

However, the latter authors ignored that a spacetime is not only defined by a metric, but also by the topology of the corresponding manifold. And in fact, as we shall explain in details later, while the Schwarzschild manifold is homeomorphic to $\mathbb{R} \times]0, \infty[\times S^2$, leaving no room for a black hole and dispensing a procedure of maximal extension, the topology of the Hilbert-Droste manifold is homeomorphic to $\mathbb{R} \times (]0, \infty[- \{\mu\}) \times S^2$ (for some real $\mu > 0$), being consequently a *different* solution of the Einstein equation. We remark that the latter solution having a disconnected manifold require a maximal extension in order to become a satisfactory spacetime (cf. Definition 20).

This was recognized by N. Stavroulakis in his writings entitled “*Mathématiques et trous noirs*” (cf. ref. [5]), which appeared in the *Gazette des mathématiciens*, and “*Vérité scientifique et trous noirs*” (cf. refs. [6]–[9]), published just four years before the Antoci & Liebscher editorial note. (We shall comment briefly on Stavroulakis’s articles in the final section). Another author, who seems to be one of the first to advocate that the solutions of Schwarzschild and of Hilbert-Droste are really different, was L. Abrams, publishing about the subject already in 1979 (cf. ref. [20]).

It is important to remark that because the Hilbert-Droste solution has a disconnected topology (which as we will show below, is not the case of the manifold in Schwarzschild’s solution), the Relativity community was lead to the “Maximal Extension” research programme, which grown from a J. Synge’s letter to the editor in a Nature’s volume which dates from 1949, and culminated in the Kruskal-Szekeres spacetime and in the Fronsdal imbedding of the Hilbert-Droste manifold – a procedure which was based in a work of E. Kasner from 1921 (almost four decades before Fronsdal’s paper was published). This, of course, inaugurated the physics of black holes.

Our paper revisit this issues from a mathematically rigorous standpoint and is organized as follows. In Section 2, we present the mathematical formalism which will be adopted in rest of our work. In particular, we discuss the warped product of manifolds, which is a powerful tool in constructing spacetimes in General Relativity, some issues concerning the extension of manifolds (which is complemented by the Appendix A) and the properties of null (or lightlike) geodesics which are useful in verifying that a given manifold is maximal. In Section 3, we set a framework in which both the Schwarzschild and the Hilbert-Droste solutions can be constructed, in such a way that a parallel between their derivations and the origin of their topological differences will be shown.

In Section 4, motivated by the disconnectedness of the Hilbert-Droste solution, we begin the search for its maximal extension, covering details normally omitted by the present literature leading to the Kruskal-Szekeres spacetime. A *brief* summary of the relevant historical developments is then presented. Lastly, we proceed to discuss the works of Kasner and Fronsdal that culminated in the

embedding of the Hilbert-Droste spacetime in a 6-dimensional vectorial manifold, thus ending this chapter in the history of General Relativity.

Finally, in Section 5, we restate our main conclusions and comment on some works in the literature. And, in Appendix B, we give a short but rigorous discussion of the Einstein-Rosen Bridge and some of its mathematical relations to the Horizon that belongs to the Kruskal-Szekeres spacetime.

2 Mathematical Formalism

In order to fix our notation and refresh the memory, we review in Subsections 2.1 and 2.2 some elementary facts concerning pseudo-Riemannian geometry, Minkowski vector spaces and spacetimes.

Then, the following two subsections are dedicated to a discussion of the *warped product*, a powerful tool that can be employed in the construction of some spacetimes in General Relativity. As we shall see, its use has at least two advantages: it can elegantly simplify calculations related to geometric quantities, as the Ricci curvature tensor, and even more important, when a spacetime is given in the form of a warped product, its manifold topology is stated without ambiguities since the beginning.

Finally, in the Subsection 2.5, we discuss some properties of null geodesics which shall be useful (cf. Section 4) in our construction of the maximal extension of the Hilbert-Droste solution (the Kruskal-Szekeres spacetime), a subject which is normally treated very informally in the current literature.

In Appendix A, our discussion of the extension of manifolds is continued from a topological point of view. There, we discuss some topological issues which may arise when two topological spaces are glued together through a continuous identification of its topological subspaces. That Appendix is however unnecessary for our main developments, but will be used in the rigorous construction of the Einstein-Rosen bridge presented in Appendix B.

2.1 Manifolds and Exponential Mapping

First, recall that

Definition 1 *A pseudo-Riemannian manifold is an ordered pair (M, g) , where M is a smooth manifold and $g \in \sec T_2^0 M$ is a metric tensor, i.e., a symmetric and non degenerate 2-covariant tensor field in M with the same index in all tangent spaces of M . We may say that M have a pseudo-Riemannian structure.*

Remember that the *index* of a symmetric bilinear form g is the greatest integer v such that there is a subspace W with the properties: $\dim W = v$ and $g(x, x) < 0$ for all $x \in W$.

When there is no fear of confusion, we may refer to a pseudo-Riemannian manifold (M, g) just by M .

Definition 2 Let M be a pseudo-Riemannian manifold and let γ be a curve from $I \subset \mathbb{R}$ into M . Let \hat{D}_γ be the induced Levi-Civita connection (of g) on γ . So we will call γ a geodesic if $\hat{D}_\gamma \gamma'(t) = 0$ for all $t \in I$.

In what follows, unless we use the adjective *segmented*, all geodesics are defined on a interval which contains $0 \in \mathbb{R}$.

Recall that a geodesic γ defined on $I \subset \mathbb{R}$ is called inextendible if and only if, for all geodesics σ defined on $J \subset \mathbb{R}$ such that $\sigma'(0) = \gamma'(0)$, we have that $J \subset I$. To each $x \in T_p M$, we will denote by γ_x the unique inextendible geodesic such that $\gamma'_x(0) = x$.

The idea of approximate the neighborhood of a point in a manifold through the tangent space in that point can be made precise by using the exponential mapping:

Definition 3 Let M be a pseudo-Riemannian manifold and let $p \in M$. Let D_p be the subset of $T_p M$ such that, for all $x \in D_p$, the domain of γ_x contains $[0, 1] \subset \mathbb{R}$. The exponential mapping \exp_p at p is the mapping from D_p into M such that $x \rightarrow \exp_p(x) = \gamma_x(1)$.

Remark 4 Let γ be a geodesic with induced Levi-Civita connection \hat{D}_γ . As, in coordinates, $\hat{D}_\gamma \gamma'(t) = 0$ corresponds to a system of ordinary differential equations of second order, the solution depends smoothly on the initial values. Then the exponential mapping is a well-defined smooth mapping.

In this paragraph, to each $\theta \in T_p^* M$, we will denote by $d\theta$ the differential mapping of θ as being a function from $T_p M$ into \mathbb{R} , and *not* the exterior derivative of θ as being a covector field. In the proof of the following Lemma, given $x \in M$, the natural homomorphism ϕ between $T_x(T_p M)$ and $T_p M$ is the mapping such that, for all covector $\theta \in T_p^* M$, $\theta[\phi(v_x)] = d\theta(v_x)$, for all $x \in T_p M$.

Lemma 5 Let M be a pseudo-Riemannian manifold. For each $p \in M$, there is a neighborhood $V \subset T_p M$ of $0 \in T_p M$ such that $\exp_p|_V$ is a diffeomorphism.

Proof. Let ϕ be the natural homomorphism between $T_x(T_p M)$ and $T_p M$. Let $v_0 \in T_0(T_p M)$, let $v = \phi(v_0)$ and let $\lambda(t) = vt$ be a mapping from \mathbb{R} into $T_p M$. So, as $\lambda'(0) = v_0$,

$$\exp_{p*}(v_0) = \exp_{p*}[\lambda'(0)] = (\exp_{p*} \circ \lambda)'(0) = v$$

Hence \exp_{p*} is the natural homomorphism ϕ . By Remark 4 and the inverse mapping theorem, the result follows. ■

Definition 6 Let M be a pseudo-Riemannian manifold and let $p \in M$. A neighborhood U of p will be called *normal* if there is a neighborhood $V \subset T_p M$ of $0 \in T_p M$ such that $\exp_p|_V$ is a diffeomorphism between V and U and, for all $x \in V$, $\{tx : t \in [0, 1] \subset \mathbb{R}\} \subset V$.

So the last Lemma ensures that we can always find a normal neighborhood for a given point.

Lemma 7 *Let M be a pseudo-Riemannian manifold, let $p \in M$ and let U be a normal neighborhood of p . So, for all $q \in U$, there is a unique geodesic γ_{pq} from $[0, 1] \subset \mathbb{R}$ into U such that $\gamma_{pq}(0) = p$, $\gamma_{pq}(1) = q$ and $\gamma'_{pq}(0) = \exp_p^{-1}(q)$.*

Proof. Let $v = \exp_p^{-1}(q)$ and let $\lambda(t) = vt$ be a mapping from \mathbb{R} into $T_p M$. Let $\sigma(t) = \exp_p \circ \lambda(t)$ be a mapping from $[0, 1] \subset \mathbb{R}$ into U . By the hypothesis on V , σ is well-defined, and by the Definition 3, σ is a geodesic. But

$$\sigma'(0) = (\exp_{p*} \circ \lambda)'(0) = \exp_{p*} [\lambda'(0)] = v$$

by the proof of the last Lemma. Hence the existence assertion. The proof of the uniqueness will be left as an easy exercise. ■

Let γ be a curve from $[a, b] \subset \mathbb{R}$ into a pseudo-Riemannian manifold M . We will say that γ is a broken-geodesic if there is a partition $(J_i)_{i \in F \subset \mathbb{N}}$ of $[a, b]$ such that each restriction $\gamma|_{J_i}$, for $i \in F$, is a segmented geodesic. In this case, we say that $\gamma(a)$ and $\gamma(b)$ are connected by a broken-geodesic.

Corollary 8 *A pseudo-Riemannian manifold M is connected if and only if, for all points $p, q \in M$, there exists a broken-geodesic γ defined on $[a, b] \subset \mathbb{R}$ such that $\gamma(a) = p$ and $\gamma(b) = q$.*

Proof. Let S be the subset of M of all points that can be connected by a broken-geodesic and let $p \in M$. Let U be a normal neighborhood of p . So, by Lemma 7, if $p \in S$, $U \subset S$. But if $p \notin S$, then $U \cap S = \emptyset$, and M cannot be connected. Hence the result. ■

In what follows, we will call a neighborhood U in a pseudo-Riemannian manifold *convex* if U is a normal neighborhood for all $p \in U$. To see a proof that a convex neighborhood always exists around any given point, see Chapter 5 of [12].

2.2 Spacetimes

Spacetimes are the manifolds upon which the General Relativity Theory is established. To define them, we need to recall some facts about Lorentz vector spaces:

Definition 9 *A Lorentz vector space is an ordered pair (V, g) , where V is a finite-dimensional linear space with dimension $\dim V \geq 2$ and g is a symmetric and non degenerate bilinear form on V with index 1.*

A sequence $(e_i)_{i \in F \subset \mathbb{N}}$ of vectors in a given Lorentz vector space (V, g) will be called orthonormal if $|g(e_i, e_j)| = \delta_{ij}$, where δ_{ij} is the Kronecker delta (i.e., $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$).

Lemma 10 *Let (V, g) be a Lorentz vector space. So there is an orthonormal basis for V .*

Proof. (i) As g is non degenerate, there is a $x \in V$ such that $g(x, x) \neq 0$. (ii) If $(e_i)_{i \in [1, k]}$ is a sequence of orthonormal vectors (for some $k < \dim V$), there is a vector e_{k+1} such that $(e_i)_{i \in [1, k+1]}$ is also orthonormal, by (i) and by the fact that g is non degenerated in the subspace $\{x \in V : g(x, e_i) = 0, i \in [1, k] \subset \mathbb{N}\}$. The result follows then by induction. ■

Definition 11 *Let (V, g) be a Lorentz vector space. A vector $x \in V$ will be called *timelike* if $g(x, x) < 0$, *spacelike* if $g(x, x) > 0$ and *null (or lightlike)* if $g(x, x) = 0$. A vector is *causal* if it is timelike or null. A subspace $W \subset V$ is called *timelike*, *spacelike* or *null* if all vectors in W are timelike, spacelike and null, respectively.*

On what follows, given a Lorentz vector space (V, g) , the orthogonal complement of $x \in V$ is the subset $x^\perp = \{z \in V : g(x, z) = 0\}$. The reader may prove that x^\perp is, in fact, a subspace.

Let $(e_i)_{i \in [1, n]}$ be an orthonormal basis for a n -dimensional Lorentz vector space (V, g) and let $(\varepsilon_i)_{i \in [1, n]}$ be a sequence numbers such that $g(e_i, e_j) = \varepsilon_i \delta_{ij}$. For the proof of the next Lemma, recall [11] that the Sylvester Theorem ensures that there is one and only one $k \in [1, n] \subset \mathbb{N}$ such that $\varepsilon_k = -1$.

Lemma 12 *Let (V, g) be a Lorentz vector space and let $x \in V$. So x^\perp is timelike (respectively, spacelike) if x is spacelike (respectively, timelike).*

Proof. Let $n = \dim V$ and suppose that x is timelike. By the proof Lemma 10, there is an orthonormal sequence $(e_i)_{i \in [1, n-1]}$ of vectors in V such that $(e_i)_{i \in [1, n]}$ is an orthonormal basis for V , where $e_n = x / \sqrt{g(x, x)}$. Let $y \in x^\perp$. So there is a sequence $(a_i)_{i \in [1, n]}$ of real numbers such that $y = \sum_{i \in [1, n]} a_i e_i$. By hypothesis, $a_n = 0$. Hence, by Sylvester Theorem, $g(y, y) = \sum_{i \in [1, n]} (a_i)^2 > 0$, i.e., y is spacelike, and the proof is analogous if x is spacelike. ■

From now on, the set of all timelike vectors in a given Lorentz vector space (V, g) will be denoted by τ , while that the set of all null vectors will be denoted by Λ . These are normally called, respectively, the *timecone* and the *lightcone* of V . The union $\tau \cup \Lambda$ will be called the *causalcone* and denoted by Υ .

Exercise 13 *Using Lemma 12, prove that the timecone, lightcone and the causalcone of a given Lorentz vector space have two disjoint components. Also, prove that the closure of a component of the timecone is a component of the lightcone. (For details, see Chapter 5 of [12] or Chapter 1 of [14]).*

Then we shall denote by τ^+ and τ^- the disjoint components of the timecone τ , and by Λ^+ and Λ^- their respective boundaries (which are, of course, the disjoint components of Λ). The closure of τ^+ and τ^- , which will be denoted by Υ^+ and Υ^- , respectively, are the components of the causalcone Υ .

Lemma 14 *Let (V, g) be a Lorentz vector space and let $x \in \tau^+$. So $y \in \Upsilon^+$ if and only if $g(x, y) < 0$ and $z \in \Upsilon^-$ if and only if $g(x, z) > 0$.*

Proof. Let f be the continuous mapping from Υ into $\mathbb{R} - \{0\}$ such that $v \rightarrow f(v) = g(x, v)$. As $f(x) < 0$ and Υ^+ is connected, $f(\Upsilon^+) = (-\infty, 0) \subset \mathbb{R}$. If $z \in \Upsilon^-$, thus $-z \in \Upsilon^+$, hence the result. ■

Now, we are ready to generalize this to a manifold:

Definition 15 *A Lorentzian manifold is an orientable 4-dimensional pseudo-Riemannian manifold whose index of the metric is 1.*

Given a Lorentzian manifold M , let π be the natural projection from TM onto M . An element $x \in TM$ will be called timelike, spacelike and null (or lightlike) if x , as an element of the Lorentz vector space $T_{\pi(x)}M$, is timelike, spacelike or null, respectively. As before, $x \in TM$ is causal if it is timelike or null.

Let γ be a curve from $I \subset \mathbb{R}$ into a pseudo-Riemannian manifold M and let \hat{D}_γ be the induced Levi-Civita connection on γ . For the proof of the following Lemma, remember that, given a vector field $X \in \sec T\gamma$ over γ , we say that X is parallel if $\hat{D}_\gamma X = 0$. Let $x \in T_{\gamma(a)}M$ for some $a \in I$. By the theory of differential equations, there is one and only one parallel vector field $X \in \sec T\gamma$ such that $X_a = x$. In this case, $y \in T_{\gamma(b)}M$ (for some $b \in I$) will be called the parallel transport (from a to b) of x along γ if $X_b = y$.

Lemma 16 *Let M be a connected Lorentzian manifold. The subset $\tau(M) \subset TM$ of all causal vectors is connected or have two components.*

Proof. Let $p \in M$ and let A be the set of all broken-geodesics in M . By the Corollary 8 and the axiom of choice, there is a mapping δ_p from M into A such that each $\delta_p(q)$ is a broken-geodesic from p into q . Let Υ_r^+ and Υ_r^- be the components of the causalcone $\Upsilon_r \subset T_r M$ for any $r \in M$. To each $q \in M$, define $\hat{\Upsilon}_q^+, \hat{\Upsilon}_q^- \subset T_q M$ to be such that $\hat{x} \in \hat{\Upsilon}_q^+$ and $\hat{y} \in \hat{\Upsilon}_q^-$ if and only if there exists $x \in \Upsilon_p^+$ and $y \in \Upsilon_p^-$ such that \hat{x} and \hat{y} are, respectively, the parallel transport of x and y along $\delta_p(q)$. Hence, by virtue of the Levi-Civita connection, $\hat{x} \in \Upsilon_q^+$ and $\hat{y} \in \Upsilon_q^-$, and $\tau(M) = \cup_{q \in M} (\hat{\Upsilon}_q^+ \cup \hat{\Upsilon}_q^-)$. Consequently, $\tau(M)$ have at most two components, and the result follows. ■

Problem 17 *Let M be a connected Lorentzian manifold and let $X \in \sec TM$ be a smooth timelike vector field, i.e., $X_p \in T_p M$ is a timelike vector for all $p \in M$. So the subset $\tau(M) \subset TM$ of all causal vectors have two components.*

Solution 18 *Let g be the metric of M and let f be the continuous mapping from $\tau(M)$ onto $\mathbb{R} - \{0\}$ such that $V_p \rightarrow f(V_p) = g(X_p, V_p)$. As $f(X_p) < 0$ for all $p \in M$, $f^{-1}(-\infty, 0)$ and $f^{-1}(0, \infty)$ must be two disconnected components, and the result follows from Lemma 16.*

In the case of the last Problem, we usually say that a vector $Y \in TM$ is future-pointing if it is in the same component of $\tau(M)$ as X .

Finally,

Definition 19 *A connected Lorentzian manifold is time-orientable if and only if the subset $\tau(M) \subset TM$ of all causal vectors has two components.*

Definition 20 *A spacetime (in General Relativity) is a connected orientable and time-orientable Lorentzian manifold (M, g) equipped with the Levi-Civita connection D of g .*

Remark 21 *A physical motivation for the last Definition is that, if we assume that the thermodynamics holds for any process in a given spacetime, it must be possible to select a “time arrow” for the physical phenomena from the second law, given a time orientation in that spacetime.*

2.3 Product of Manifolds

In what follows, given two manifolds M and N , the natural projections π_M and π_N of $M \times N$ are the mappings from $M \times N$ into M and N , respectively, such that $\pi_M(p, q) = p$ and $\pi_N(p, q) = q$.

Lemma 22 *Let (M, g_M) and (N, g_N) be pseudo-Riemannian manifolds and let π_M and π_N be the natural projections of $M \times N$. Then $(M \times N, g)$, where*

$$g = \pi_M^*(g_M) + \pi_N^*(g_N)$$

is itself a pseudo-Riemannian manifold, called the product manifold of (M, g_M) and (N, g_N) .

The proof is a direct application of Definition 1 and will be left as an easy exercise.

In order to transport mappings, vectors and tensors from manifolds M and N to the product manifold $M \times N$, the notion of a lift will be introduced below. For the sake of brevity, consider the following notation:

$$T_{(p,q)}M = T_{(p,q)}(M \times \{q\})$$

$$T_{(p,q)}N = T_{(p,q)}(\{p\} \times N)$$

for all $(p, q) \in M \times N$.

Lemma 23 *Let M and N be smooth manifolds. So to each $(p, q) \in M \times N$, $T_{(p,q)}(M \times N)$ is the direct sum of $T_{(p,q)}M$ and $T_{(p,q)}N$.*

Proof. By definition, $\pi_M|(\{p\} \times N)$ is a constant function. So $\pi_{M*}(T_{(p,q)}N) = \{0\}$. But $\pi_{M*}|T_{(p,q)}M$ is an isomorphism onto T_pM . Hence $T_{(p,q)}M \cap T_{(p,q)}N = \{0\}$. The result follows then by $\dim T_{(p,q)}(M \times N) = \dim T_{(p,q)}M + \dim T_{(p,q)}N$. ■

Because of the identifications between $T_{(p,q)}M$ and T_pM and between $T_{(p,q)}N$ and T_qN , one normally recall the last Lemma in applications as saying that $T_{(p,q)}(M \times N) = (T_pM) \times (T_qN)$.

Hereafter, given a manifold M , the set of all smooth mappings from M into \mathbb{R} will be denoted by $F(M)$.

Definition 24 *Let M and N be smooth manifolds let π_M and π_N be the natural projections of $M \times N$. We define the lifts in $M \times N$ of the mappings $f \in F(M)$ and $g \in F(N)$ to be the functions $\mathbf{f} = f \circ \pi_M$ and $\mathbf{g} = g \circ \pi_N$, respectively. We also define the lifts in $M \times N$ of the vectors $x \in T_pM$ and $y \in T_qN$ as the unique $\mathbf{x} \in T_{(p,q)}M$ and $\mathbf{y} \in T_{(p,q)}N$, respectively, such that $\pi_{M*}\mathbf{x} = x$ and $\pi_{N*}\mathbf{y} = y$.*

Remark 25 *The uniqueness assertion in the last definition is ensured by Lemma 23.*

We can extrapolate the above definition to vector fields in the following way:

Definition 26 *Let M and N be smooth manifolds. Let $X \in \sec TM$ and $Y \in \sec TN$ be a vector fields. We define the lifts in $M \times N$ of X and Y to be the unique vector fields $\mathbf{X}, \mathbf{Y} \in \sec T(M \times N)$ such that \mathbf{X}_p is the lift in $M \times N$ of $X_p \in T_pM$ and \mathbf{Y}_p is the lift in $M \times N$ of $Y_p \in T_pN$. We will say that \mathbf{X} is a horizontal lift in $M \times N$, while that \mathbf{Y} is a vertical lift.*

Remark 27 *Using coordinates, one can prove that the lift of a smooth vector field is by itself smooth.*

Example 28 *In \mathbb{R}^2 with natural coordinates (x, y) , $\frac{\partial}{\partial x}$ is the horizontal lift of $\frac{d}{dt}$, while that $\frac{\partial}{\partial y}$ is the vertical one.*

From now on, in the terminology of Definition 26, the set of all horizontal lifts in $M \times N$ will be denoted by $\mathcal{L}(M)$, whereas the set of all vertical lifts will be denoted by $\mathcal{L}(N)$.

Finally, we need to define the lift of a r -covariant tensor field:

Definition 29 *Let M and N be smooth manifolds and let π_M and π_N be the natural projections of $M \times N$. Let $A \in \sec T^r M$ and $B \in \sec T^r N$ be r -covariant tensor fields. We define the lifts in $M \times N$ of A and B to be the unique r -covariant tensor fields $\mathbf{A}, \mathbf{B} \in \sec T^r(M \times N)$ such that, for all $(p, q) \in M \times N$ and $(v_i)_{i \in [1, r]} \in T_{(p,q)}(M \times N)$, $\mathbf{A}(v_1, \dots, v_r) = A(\pi_{M*}(v_1), \dots, \pi_{M*}(v_r))$ and $\mathbf{B}(v_1, \dots, v_r) = B(\pi_{N*}(v_1), \dots, \pi_{N*}(v_r))$.*

Remark 30 (a) *Using Lemma 23, one can prove the uniqueness assertion.* (b) *This definition cannot be used to lift an arbitrary (s, r) -tensor field, since that π_M^* and π_{M*} goes in "opposite" directions. But using Definition 26, the reader is invited to inquire how to lift a $(s, 1)$ -tensor field.*

2.4 Warped Product

In General Relativity, many spacetimes can be constructed in the following way:

Definition 31 Let (B, g_B) and (F, g_F) be pseudo-Rimannian manifolds and π_M and π_N be the natural projections of $B \times F$. Let f be a smooth mapping from F into \mathbb{R}^+ (the set of positive real numbers). We define the warped product $B \times_f F$ to be the pseudo-Rimannian manifold $(B \times F, g)$ such that

$$g = \pi_B^*(g_B) + (f \circ \pi_F)^2 \pi_F^*(g_F)$$

The function f may be called the warping mapping of $B \times_f F$.

Example 32 Let r be the identity mapping in \mathbb{R}^+ and let (ϕ, φ) be polar coordinates in $S^2 = S^2(1)$. Let

$$\eta = d\phi \otimes d\phi + \sin^2 \phi d\varphi \otimes d\varphi$$

be the Euclidean metric in S^2 . Then $\mathbb{R}^+ \times_r S^2$ is isometric to the Euclidean space $\mathbb{R}^3 - \{0\}$.

Exercise 33 Let n be a positive integer and let $v \in [0, n) \subset \mathbb{N}$. Let $(x^i)_{i \in [0, n]}$ be the natural coordinates of \mathbb{R}^{n+1} . So $(\mathbb{R}^{n+1}, \zeta)$ is the pseudo-Euclidean n -space of index v when

$$\zeta = - \sum_{i \in [1, v]} dx^i \otimes dx^i + \sum_{i \in [v+1, n+1]} dx^i \otimes dx^i.$$

Then the pseudo-Euclidean n -sphere S_v^n of index v is the n -sphere $S^n \subset \mathbb{R}^{n+1}$ with the induced connection of $(\mathbb{R}^{n+1}, \zeta)$. Show how S_v^n can be written as a warped product of S^{n-v} .

Recall that, given a smooth mapping f from pseudo-Rimannian manifold M (together with a metric tensor g) into \mathbb{R} , $\text{grad}(f)$ is the vector field metric equivalent to df , that is,

$$g(\text{grad}(f), X) = df(X) = X(f)$$

for all vector $X \in TM$. Then the Hessian of f is defined to be the 2-covariant tensor field such that

$$(V, W) \rightarrow H^f(V, W) = VW(f) - (D_V W) = g(D_V(\text{grad}(f)), W)$$

and the Laplacian of f is simply the contraction of H^f , i.e., $\Delta(f) = CH^f$.

The following Lemma will be our bridge between the geometry of B and F and its warped product $B \times_f F$:

Lemma 34 *With the notation of Definition 31, let $(M, g_M) = B \times_f F$, let Ric^M be the Ricci curvature tensor of M and let $\mathbf{Ric}^B \in \mathcal{L}(B)$ and $\mathbf{Ric}^F \in \mathcal{L}(F)$ be the lifts of the Ricci tensors of B and F , respectively. Suppose that $\dim F > 1$ and define the mapping*

$$\mathfrak{I}(f) = \frac{\Delta(f)}{f} + (\dim F - 1) \frac{g_M(\text{grad}(f), \text{grad}(f))}{f^2}$$

from M into \mathbb{R} . Hence, for all $\mathbf{X}, \mathbf{Y} \in \mathcal{L}(B)$ and $\mathbf{V}, \mathbf{W} \in \mathcal{L}(F)$,

$$Ric^M(\mathbf{X}, \mathbf{W}) = 0,$$

$$Ric^M(\mathbf{V}, \mathbf{W}) = \mathbf{Ric}^F(\mathbf{V}, \mathbf{W}) - \mathfrak{I}(f)g_M(\mathbf{V}, \mathbf{W}),$$

$$Ric^M(\mathbf{X}, \mathbf{Y}) = \mathbf{Ric}^B(\mathbf{X}, \mathbf{Y}) - \frac{\dim F}{f} H^f(\mathbf{X}, \mathbf{Y}).$$

The proof of this Proposition follows a tedious application of definitions and will then be omitted. The interested reader may consult the Chapter 7 of [12].

2.5 Null Geodesics and Maximal Extensions

Definition 35 *A pseudo-Riemannian manifold M will be called maximal when, for all pseudo-Riemannian manifolds N with the same dimension of M for which M is isometric to an open submanifold, $M = N$.*

Differently from the Riemannian case, we cannot use the Hopf-Rinow Theorem to decide when our spacetime is maximal. However, we can do it by studying the behavior of the null geodesics.

Lemma 36 *Let M be a spacetime and let U be a convex neighborhood in M . So for all points $p, q \in U$, there is one $r \in U$ such that the unique geodesics from p into r and from r into q are nulls.*

To prove this, we will use the Gauss Lemma and introduce some terminology first.

For the last of this section, let (M, g) be a pseudo-Riemannian manifold, let $p \in M$, let $x \in T_p M$ and let ϕ_x be the natural homomorphism between $T_x(T_p M)$ and $T_p M$ (recall the comment above Lemma 5). In what follows, a vector $v \in T_x(T_p M)$ will be called radial if there is a real $k \neq 0$ such that $\phi(v) = kx$, and we will denote just by g the metric for both $T_p M$ and $T_x(T_p M)$.

Lemma 37 (Gauss Lemma) *Let (M, g) be a pseudo-Riemannian manifold and let $p \in M$. Let $v, w \in T_x(T_p M)$ and suppose that v is radial. Then*

$$g(v, w) = g(\exp_{p*} v, \exp_{p*} w).$$

Proof. Let $\lambda(t, r) = t[\phi_x(v) + s\phi_x(w)]$ be a mapping from $\mathbb{R} \times \mathbb{R}$ into $T_p M$ and let $x(t, r) = \exp_p \circ \lambda(t, r)$ be a mapping from $\mathbb{R} \times \mathbb{R}$ into U . As $(D_1 \lambda)(1, 0) = v$ and $(D_2 \lambda)(1, 0) = w$, we have

$$(D_1 x)(1, 0) = \exp_{p*} v \quad (D_2 x)(1, 0) = \exp_{p*} w.$$

But, by the definition of the exponential mapping, $t \mapsto x(t, r)$ is a geodesic. Hence $D_1^2 x = 0$ and $g(D_1 x, D_1 x) = g(\phi_x(v) + s\phi_x(w), \phi_x(v) + s\phi_x(w))$. Thus

$$D_1 g(D_1 x, D_2 x) = g(D_1 x, D_2 D_1 x) = \frac{1}{2} D_2 g(D_1 x, D_1 x) = g(\phi_x(w), \phi_x(v) + s\phi_x(w)),$$

which implies

$$[D_1 g(D_1 x, D_2 x)](t, 0) = g(\phi_x(v), \phi_x(w)).$$

The result follows then from the fact that $g(D_1 x(0, 0), D_2 x(0, 0)) = 0$ and an elementary calculation. ■

From now on, the *position vector field* $P \in \sec T(T_p M)$ in $T_p M$ is defined to be the vector field such that $P_x = \phi_x^{-1}(x)$, and the *quadratic form* Q_p in $T(T_p M)$ is the mapping into \mathbb{R} given by $Q_p(x) = g(x, x)$. Then we may write that $Q_p = g(P, P)$.

Exercise 38 Let \tilde{D} be the Levi-Civita connection on the vector space $T_p M$. Prove that, if P is the position vector field, then $\tilde{D}_v P = v$ for all $v \in T(T_p M)$. (Hint: if you feel lost, appeal to coordinates).

Lemma 39 Let M be a pseudo-Riemannian manifold and let $p \in M$. Let P and Q be the position vector field and the quadratic form in $T_p M$, respectively. So

$$\text{grad } Q = 2P$$

Proof. Let $v \in T_x(T_p M)$ for some $x \in T_p M$. Then:

$$g(\text{grad } Q, v) = dQ(v) = v[g(P, P)] = 2g(P, v)$$

by the last exercise, and the proof is over. ■

The exponential mapping can extend the position vector field and the quadratic form over a normal neighborhood in the following way. We define the (transported) *position vector field* $\mathbf{P} \in \sec TU$ to be the vector field over U given by $\mathbf{P} = \exp_{p*} P$, and the (transported) *quadratic form* \mathbf{Q} to be the mapping on U such that $x \rightarrow \mathbf{Q}(x) = Q \circ \exp_p^{-1}(x)$.

Lemma 40 Let U be a normal neighborhood in a given pseudo-Riemannian manifold M and let \mathbf{P} and \mathbf{Q} be the transported position vector field and the transported quadratic form, respectively. So

$$\text{grad } \mathbf{Q} = 2\mathbf{P}.$$

Proof. Let $y \in T_q U$ for some $q \in U$. Then

$$g(\text{grad } \mathbf{Q}, y) = d(Q \circ \exp_p^{-1})(y) = g(\text{grad } Q, \exp_p^{-1} y) = 2g(P, \exp_p^{-1} y)$$

and the result follows by the Gauss Lemma. ■

Proof of Lemma 36. Let Υ_q^+ and Υ_q^- be the disjoint components of the causal cone of $T_q M$. As U is a convex neighborhood, there is a unique geodesic σ from p into q . Without loss of generality, suppose that $\sigma'(0) \in \Upsilon_q^-$ (or in intuitive terms, p is in the past of q). Let γ be a null geodesic defined on $I \subset \mathbb{R}$ such that $\gamma(0) = p$. Let \mathbf{P} and \mathbf{Q} be the transported position vector field and the transported quadratic form in $T_q M$, respectively. So

$$(\mathbf{Q} \circ \gamma)'(t) = d\mathbf{Q}[\gamma'(t)] = 2g(\mathbf{P}_{\gamma(t)}, \gamma'(t)).$$

But $\mathbf{Q} \circ \gamma(0) = \mathbf{Q}(p) \geq 0$, by hypothesis. If $\mathbf{Q} \circ \gamma(0) = 0$, the result follows trivially. Then suppose that $\mathbf{Q} \circ \gamma(0) > 0$. By Lemma 14 and by the Gauss Lemma, the equation above shows that $(\mathbf{Q} \circ \gamma)'(t) < 0$. Hence, there is a $k \in I$ such that $\mathbf{Q} \circ \gamma(k) = 0$. Then let $r = \gamma(k)$ and the proof is over. ■

Lemma 41 *Let M be a spacetime and let N be an open submanifold of M with the induced connection. Assume that, if γ is a null geodesic from $I \subset \mathbb{R}$ into M such that $\gamma(I) \cap N \neq \emptyset$, then $\gamma(I) \subset N$. Hence $M = N$.*

Proof. Suppose that $M \neq N$ and let U be a convex neighborhood in M such that $U \cap \partial N \neq \emptyset$. By hypothesis, there are $p \in U - N$ and $q \in U \cap N$. By the Lemma 36, there is some $r \in U$ such that the unique geodesics γ_{pr} from p into r and γ_{rq} from r into q are nulls. By hypothesis, γ_{pr} lies on N . Hence $r \in N$. So γ_{rq} lies on N . Thus $q \in N$. Contradiction. ■

Remark 42 *Physically, the last Corollary means that a spacetime is maximal if one cannot “see” beyond it.*

3 Schwarzschild and Hilbert-Droste Solutions

It is well-known that K. Schwarzschild [16] was the first to find the exact gravitational field of a point of mass in General Relativity. However, a year later, the same problem was differently approached by D. Hilbert [17] and J. Droste [18], differences which will be discussed below.

In Section 3.1, we shall build a “spacetime” model in which the Schwarzschild and Hilbert-Droste are particular cases. Hence, we show in Section 3.2 how to generate solutions from such a model and we illustrate with a simple example.

Finally, we present in the last two sections the derivation of the Hilbert-Droste and Schwarzschild solutions and we finish by discussing if these are actually the same or not.

3.1 Building the Model

Let (t, r) be the natural coordinates of \mathbb{R}^2 and let $P \subset \mathbb{R} \times \mathbb{R}^+$ be an open submanifold. In what follows, (t, h) will be called *special coordinates* of P if and only if there is a diffeomorphism ϕ from $r(P)$ into \mathbb{R} such that $h = \phi \circ r$.

Definition 43 A Schwarzschild model is an ordered list $(P, (t, h), f, g, \alpha)$, where $P \subset \mathbb{R} \times \mathbb{R}^+$ is an open submanifold, (t, h) is some special coordinates of P and f, g, α are smooth mappings from $h(P)$ into \mathbb{R}^+ such that

$$\lim_{h \rightarrow \infty} f(h) = \lim_{h \rightarrow \infty} g(h) = 1$$

Definition 44 Let $M = (P, (t, h), f, g, \alpha)$ be a Schwarzschild model. We define the corresponding Schwarzschild plane Π_M to be the pseudo-Riemannian manifold (P, ζ) such that

$$\zeta = -(f \circ h)dt \otimes dt + (g \circ h)dh \otimes dh$$

In building a manifold through the warped product, the first step is to study the geometry of its parts. In our case, we start by

Lemma 45 Given a Schwarzschild model $M = (P, (t, h), f, g, \alpha)$, let D be the Levi-Civita connection of its Schwarzschild plane Π_M . Thus

$$\begin{aligned} D_{\partial_t} \partial_t &= \frac{f'(h)}{2g(h)} \frac{\partial}{\partial h} & D_{\partial_h} \partial_h &= \frac{g'(h)}{2g(h)} \frac{\partial}{\partial h} \\ D_{\partial_t} \partial_h &= D_{\partial_h} \partial_t = -\frac{f'(h)}{2f(h)} \frac{\partial}{\partial t} \end{aligned}$$

Proof. As the dimension of P is 2, a direct computation is viable. So let $(\Gamma_{i,j}^k)_{(k,i,j) \in [1,2]^3}$ be the Christoffel symbols. We use the well known equation

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{m \in [1,2]} \eta^{km} \left(\frac{\partial \zeta_{im}}{\partial x^j} + \frac{\partial \zeta_{jm}}{\partial x^i} - \frac{\partial \zeta_{ij}}{\partial x^m} \right)$$

where $\zeta_{ij} = \zeta(\partial/\partial x^i, \partial/\partial x^j)$ and $x^1 = t, x^2 = h$. Thus, for example,

$$\begin{aligned} \Gamma_{11}^1 &= 0 \\ \Gamma_{11}^2 &= -\frac{1}{2} \zeta^{22} \frac{\partial \zeta_{11}}{\partial h} = \frac{f'(h)}{2g(h)} \end{aligned}$$

and the identity for $D_{\partial_t} \partial_t$ follows. The last two will be left as an easy exercise. ■

So now we define our spacetime model:

Definition 46 Let S^2 be the Euclidean 2-sphere and let $M = (P, (t, h), f, g, \alpha)$ be a Schwarzschild model. So the (Schwarzschild-like) spacetime \mathbf{S}_M associated with M is the warped product

$$\Pi_M \times_{\alpha} S^2$$

Remark 47 *As a Schwarzschild-like spacetime \mathbf{S} is a pseudo-Riemannian manifold, it can have distinct representations as a Schwarzschild model. Indeed, to each possible choice of special coordinates (t, h) of P , there are mappings f, g, α such that $M = (P, (t, h), f, g, \alpha)$ implies $\mathbf{S} = \mathbf{S}_M$. The submanifold P of $\mathbb{R} \times \mathbb{R}^+$ (see Definition 44) is, in the other hand, fixed: it is a part of the manifold of \mathbf{S} . We only introduced the notion of a “Schwarzschild model” because, in finding a solution to Einstein equation (see next section), it is important to keep a track of the coordinate system which we are using.*

Remark 48 (a) *The spacetime in Definition 46 can be time oriented by lifting the coordinate vector $\partial/\partial t$; for more in time orientability, see Chapter 5 of [12].* (b) *The traditional physical motivations for the last Definition are that its corresponding spacetime is “static” with respect to the “time” t (see Chapter 12 of [12] for a rigorous definition), spherically symmetric and, as $h \rightarrow \infty$, Π_M approach the Minkowski “plane” (see Chapter 1 of [14]).*

3.2 Generating Solutions

Recall that a spacetime obeys the Einstein field equation in vacuum if and only if it is Ricci flat.

In the following Proposition we will use Lemma 34 to find the restrictions that the Einstein equation imposes upon our spacetime model:

Proposition 49 *Let $M = (P, (t, h), f, g, \alpha)$ be a Schwarzschild model. Its spacetime \mathbf{S}_M satisfies the Einstein field equation in vacuum if and only if*

$$K = \frac{\alpha'(r)f'(r)}{\alpha(r)f(r)g(r)} = \frac{2}{\alpha(r)g(r)} \left[\alpha''(r) - \frac{g'(r)}{2g(r)}\alpha'(r) \right]$$

$$\mathfrak{S}(\alpha) = \frac{1}{[\alpha(h)]^2}$$

where K is the sectional curvature of the Schwarzschild plane of M , given by

$$K = -\frac{1}{2\sqrt{f(r)g(r)}} \left[\frac{f'(r)}{\sqrt{f(r)g(r)}} \right]'$$

and

$$\mathfrak{S}(\alpha) = \frac{1}{\alpha(h)} \left\{ \frac{\alpha''(h)}{g(h)} + \frac{\alpha'(h)}{2g(h)} \left[\frac{f'(h)}{f(h)} - \frac{g'(h)}{g(h)} \right] + \frac{[\alpha'(h)]^2}{g(h)\alpha(h)} \right\}$$

In the proof of this Proposition, the following two Lemma will be used:

Lemma 50 *Let (M, g) be a pseudo-Riemannian surface (that is, a pseudo-Riemannian manifold such that $\dim M = 2$). Let Ric be its Ricci curvature tensor and let K be its sectional curvature. Then*

$$Ric = Kg$$

Proof. Let (u, v) be orthogonal coordinates in some neighborhood of M (which always exists since we can employ a frame field; see, e.g., Chapter 3 of [12]) and let R be the Riemannian curvature tensor of (M, g) . Remember that, if $x, y \in T_p M$ are linearly independent vectors (for some $p \in M$),

$$K(x, y) = \frac{g(R_{xy}x, y)}{Q(x, y)}$$

where

$$Q(x, y) = g(x, x)g(y, y) - g(x, y)^2$$

Since M has dimension 2, K is a smooth mapping in $F(M)$. But by definition

$$\begin{aligned} Ric(x, x) &= \frac{g(R_{x.\partial_u}x, \partial_u)}{g(\partial_u, \partial_u)} + \frac{g(R_{x.\partial_v}x, \partial_v)}{g(\partial_v, \partial_v)} \\ &= K_p \left[\frac{Q(x, \partial_u)}{g(\partial_u, \partial_u)} + \frac{Q(x, \partial_v)}{g(\partial_v, \partial_v)} \right] \end{aligned}$$

Then the result follows by a direct substitution in the above identity, and the details are left as an easy exercise. ■

As usual, in the following Lemma the partial derivative $\partial f / \partial x$ of a mapping f will be denoted just by f_x .

Lemma 51 *Let (M, g) be a pseudo-Riemannian surface with sectional curvature K . Let (u, v) be an orthogonal coordinate system over M , let $e, g \in \sec F(M)$ be positive real-valued mappings and let $\varepsilon_1^2 = \varepsilon_2^2 = 1$ be real numbers such that $\varepsilon_1 e^2 = g(\partial_u, \partial_u)$ and $\varepsilon_2 g^2 = g(\partial_v, \partial_v)$, where ∂_u and ∂_v are the coordinate vectors of (u, v) . Therefore*

$$K = -\frac{1}{eg} \left[\varepsilon_1 \left(\frac{e_v}{g} \right)_v + \varepsilon_2 \left(\frac{g_u}{e} \right)_u \right]$$

Proof of Proposition 49. Let Ric^{Π_M} and Ric^{S^2} be the Ricci curvature tensors of the Schwarzschild plane Π_M and of the Euclidean 2-sphere S^2 , respectively. By Lemma 34, the Einstein field equation in vacuum ($Ric = 0$) is equivalent to

$$\mathbf{Ric}^{\Pi_M}(\mathbf{X}, \mathbf{Y}) = \frac{2}{\alpha(r)} H^\alpha(X, Y)$$

$$\mathbf{Ric}^{S^2}(\mathbf{V}, \mathbf{W}) = \Im(\alpha) g(\mathbf{V}, \mathbf{W})$$

for all $V, W \in \sec TS^2$ and $X, Y \in \sec TP$, where $\mathbf{V}, \mathbf{W} \in \mathcal{L}(S^2)$ and $\mathbf{X}, \mathbf{Y} \in \mathcal{L}(P)$ are their respective lifts.

By Lemma 45 it is

$$\begin{aligned} H^\alpha(\partial_t, \partial_t) &= -\frac{f'(h)}{2g(h)} \alpha'(r), \\ H^\alpha(\partial_h, \partial_h) &= \alpha''(h) - \frac{g'(h)}{2g(h)} \alpha'(h). \end{aligned}$$

Let K be the sectional curvature of Π_M . Using Lemma 50, we find that

$$\frac{f'(h)\alpha'(h)}{f(h)g(h)\alpha(h)} = K = \frac{2}{g(h)\alpha(h)}\alpha''(h) - \frac{g'(h)}{[g(h)]^2\alpha(h)}\alpha'(h)$$

and the expression for K is a direct use of Lemma 51.

Finally, Lemma 50 gives that

$$\mathbf{Ric}^{S^2}(\mathbf{V}, \mathbf{W}) = \frac{g(\mathbf{V}, \mathbf{W})}{[\alpha(r)]^2}$$

Hence the second equation of the Proposition follows. The last is only a direct computation, and will be left as an exercise (for the Definition of $\mathfrak{S}(\alpha)$, see Lemma 34). ■

Problem 52 Fix the submanifold $P \subset \mathbb{R} \times \mathbb{R}^+$. (a) Is a Schwarzschild model $M = (P, (t, h), f, g, \alpha)$ uniquely determined by Proposition 49? (b) Is the associated spacetime \mathbf{S}_M of M , which satisfies Einstein equation in vacuum, uniquely determined?

Solution 53 (a) No. (b) Yes. One can add to the Einstein field equation some “coordinate condition” in order to determine f, g, α uniquely. After this, we have a pseudo-Riemannian manifold (in particular, a spacetime) whose metric and (if P was given) topology is well-defined. If we use some different “coordinate condition”, we must find another set of $\mathbf{f}, \mathbf{g}, \alpha$, but this is because we are using distinct coordinate systems (see Remark 47).

Indeed, the reader must already know the “Schwarzschild” solution which is normally presented in the current literature (see the next section). We illustrate in the following example another possible choice for f, g, α which also satisfies Proposition 49 and the conditions of Definition 43:

Example 54 Let $\alpha(h) = r + \mu$. Assume that the spacetime \mathbf{S}_M of a Schwarzschild model $M = (P, (t, h), f, g, \alpha)$ satisfies Einstein equation in vacuum. So by the first equation in Proposition 49,

$$[f(h)g(h)]' = 0$$

But by Definition 43 (recall the limit conditions), $f(r)g(r) = 1$. Then by the second pair of equations of the same Proposition, one finds that

$$g'(h) = \frac{g(h)[1 - g(h)]}{r + \mu}$$

Solving this equation, we find as a possible solution

$$g(h) = \frac{h + \mu}{h - \mu}$$

Of course that $\lim_{r \rightarrow \infty} g(r) = 1$, as we needed. Hence, in terms of coordinates, the metric \mathbf{S}_M reads:

$$-\frac{h-\mu}{h+\mu}dt \otimes dt + \frac{h+\mu}{h-\mu}dr \otimes dr + (r+\mu)^2 \zeta_{S^2}$$

where ζ_{S^2} is the Euclidean metric of S^2 .

3.3 Hilbert-Droste Solution

The lesson which we must take from Problem 52 and Example 54 is that, in order to find some solution of Einstein equation, one needs to impose some “coordinate condition”.

The way followed by Hilbert was very simple and elegant, and can be summarized in the following definition:

Definition 55 *A Schwarzschild model $M = (P, (t, h), f, g, \alpha)$ will be called a Hilbert model if and only if the special coordinates (t, h) of P were chosen such that $\alpha \circ h = \text{id}_{\mathbb{R}}$.*

In what follows, let (w_1, w_2, w_3, w_4) be the natural coordinates of \mathbb{R}^4 . So in Hilbert words [17] (translation from [19]),

According to Schwarzschild, if one poses

$$\begin{aligned} w_1 &= r \cos \vartheta \\ w_2 &= r \sin \vartheta \cos \varphi \\ w_3 &= r \sin \vartheta \sin \varphi \\ w_4 &= l \end{aligned}$$

the most general interval corresponding to these hypotheses is represented in spatial polar coordinates by the expression

$$(42) \quad F(r)dr^2 + G(r)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + H(r)dl^2$$

where $F(r)$, $G(r)$ and $H(r)$ are still arbitrary functions of r . If we pose

$$r^* = \sqrt{G(r)}$$

we are equally authorized to interpret r^* , ϑ and φ as spatial polar coordinates. If we substitute in (42) r^* for r and drop the symbol $*$, it results the expression

$$M(r)dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + W(r)dl^2$$

where $M(r)$ and $W(r)$ means the two essentially arbitrary functions of r .

With the last definition, we are able to derive the Hilbert-Droste *metric*:

Proposition 56 *Let $M = (P, (t, h), f, g, \alpha)$ be a Hilbert model. Its spacetime \mathbf{S}_M obeys Einstein equation in vacuum if and only if*

$$f(h) = \frac{1}{g(h)} = 1 - \frac{\mu}{h}$$

for some real μ .

Proof. By the first equation of Proposition 49, we have (like in Example 54),

$$[f(h)g(h)]' = 0.$$

But by Definition 44 (recall the limit conditions), $f(h)g(h) = 1$. Then by the second pair of equations of the same Proposition, one finds that

$$g'(h) = \frac{g(h)[1 - g(h)]}{h}$$

Hence, there is a real number μ such that

$$g(h) = \frac{1}{1 - \mu/h}$$

and the proposition is proved. ■

We do not have, however, the complete *Hilbert-Droste solution*. We only have its metric, which is just half the story. To have in hands a *proper solution*, we must set up a topology, which in this case means to pick up some $P \subset \mathbb{R} \times \mathbb{R}^+$ (recall Definitions 44 and 46).

Note that the largest submanifold of $\mathbb{R} \times \mathbb{R}^+$ in which the mappings in the last Proposition are smooth is $\mathbb{R} \times (\mathbb{R}^+ - \{\mu\})$. Then, we are motivated to state the Hilbert-Droste solution:

Definition 57 (Hilbert-Droste solution) *Given a real number μ , the Hilbert-Droste solution $H(\mu)$ is the spacetime \mathbf{S} for which there is a Hilbert model $M = (P, (t, h), f, g, \alpha)$ such that $\mathbf{S} = \mathbf{S}_M$,*

$$P = \mathbb{R} \times (\mathbb{R}^+ - \{\mu\}),$$

and

$$f(h) = \frac{1}{g(h)} = 1 - \frac{\mu}{h}.$$

So by Proposition 56, the Hilbert-Droste solution obeys the Einstein equation in vacuum.

What distinguish the coordinate expression for the metric in the above Proposition and in Example 54 is the choice of coordinates. However, are Example 54 and Definition 57 describing the same solution?

Problem 58 Let $\mu \in \mathbb{R}$ and let $M = (P, (t, h), f, g, \alpha)$ be as in Example 54. Choose $P \subset \mathbb{R} \times \mathbb{R}^+$ to be the largest submanifold for which f, g are smooth (and the corresponding metric non degenerated). Is the spacetime \mathbf{S}_M a Hilbert-Droste solution?

Solution 59 Yes. But taking into account that such a metric have a singularity in $r = \mu$ we see that the largest possible P is $\mathbb{R} \times (\mathbb{R}^+ - \{\mu\})$, hence it has the same topology as Hilbert-Droste.

Remark 60 Playing with Proposition 49, one can generate an infinite set of metrics for a Schwarzschild-like spacetime which satisfies Einstein equation. In principle, one can find a coordinate transformation which transform these metric expressions into each other. However, if we are presented with two spacetimes whose metric expressions can be transformed into each other in some coordinate chart, it does not means that they are the same solution: it is necessary to take care about the topology, which in the approach of this paper depends on a submanifold $P \subset \mathbb{R} \times \mathbb{R}^+$.

3.4 Schwarzschild Solution

In this paragraph, let (u^1, u^2, u^3, u^4) be a coordinate system on a given spacetime with metric g . When Schwarzschild found his solution in 1916, he used the following form of the Einstein field equations in vacuum

$$\sum_{k \in [1,4]} \frac{\partial \Gamma^k_{ij}}{\partial u^k} + \sum_{(k,l) \in [1,4]^2} \Gamma^k_{il} \Gamma^l_{kj} = 0,$$

$$\sqrt{-\det g} = 1,$$

for all $(i, j) \in [1, 4]^2$, where $(\Gamma^k_{ij})_{(k,i,j) \in [1,4]^3}$ are the Christoffel symbols and $\det g$ is the determinant of the matrix whose elements are $g_{ij} = g(\partial/\partial u^i, \partial/\partial u^j)$ (see [23]). The second equation is such that only unimodular coordinate transformations preserves the “mathematical form” of the field equations.

Schwarzschild started his work by setting the spacetime manifold to be $\mathbb{R} \times \{\mathbb{R}^3 - \{0\}\}$. As he wanted a spherically symmetric solution, it was natural for him to introduce spatial polar coordinates. But the transformation from the natural coordinates of \mathbb{R}^3 to polar coordinates is not, of course, unimodular. In his own words [16] (translation from [22]):

When one goes over to polar co-ordinates according to $x = r \sin \vartheta \cos \phi$, $y = r \sin \vartheta \sin \phi$, $z = r \cos \vartheta$ (...) the volume element (...) is equal to $r^2 \sin \vartheta dr d\vartheta d\phi$, [so] the functional determinant $r^2 \sin \vartheta$ of the old with respect to the new coordinates is different from 1; then the field equations would not remain in unaltered form if one would calculate with these polar co-ordinates, and one would have to perform a cumbersome transformation.

Then Schwarzschild proceeded in the following way (also from [16]):

However there is an easy trick to circumvent this difficulty. One puts:

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos \vartheta, \quad x_3 = \phi$$

Then we have for the volume element: $r^2 \sin \vartheta dr d\vartheta d\phi = dx_1 dx_2 dx_3$. The new variables are then polar co-ordinates with the determinant 1. They have the evident advantages of polar co-ordinates for the treatment of the problem, and at the same time (...) the field equations and the determinant equation remain in unaltered form.

The reader must take in mind that, in the Schwarzschild approach, the coordinate condition need to solve the equations of Proposition 49 (recall also Problem 52) must satisfies the Einstein's determinant equation.

However, thanks to the warped product, we do not need to concern with any "polar coordinates with determinant 1" here. Indeed, using the following definition, we can do the whole derivation without any mention to the coordinates of S^2 :

Definition 61 *A Schwarzschild model $M = (P, (t, h), f, g, \alpha)$ will be called a unimodular model if and only if the special coordinates (t, h) of P were chosen such that*

$$f(h)g(h) [\alpha(h)]^4 = 1$$

Remark 62 *Let ζ_{S^2} be the Euclidean metric of S^2 , let $M = (P, (t, h), f, g, \alpha)$ be a unimodular model and let g be the metric of the spacetime \mathbf{S}_M of M . In the notation of the first paragraph, if we give a coordinate expression to ζ_{S^2} such that $\det \zeta_{S^2} = 1$,*

$$-\det g = f(h)g(h) [\alpha(h)]^4 \det \zeta_{S^2} = 1$$

we have the Schwarzschild "original" coordinate condition.

To see how Definition 61 together with Proposition 49 determine the warping mapping α up to two constants, we state the following Lemma:

Lemma 63 *If the spacetime of a unimodular model $(P, (t, h), f, g, \alpha)$ satisfies the Einstein field equation in vacuum, then there are real numbers λ, μ such that*

$$\alpha(h) = \lambda (3h + \mu^3)^{1/3}$$

Proof. By hypothesis,

$$[f(h)g(h)]' [\alpha(h)] + 4 [f(h)g(h)] \alpha'(h) = 0$$

So, using the first equation of Proposition 49 we get

$$\alpha''(h) = -2 \frac{[\alpha'(h)]^2}{\alpha(h)}$$

whose solution is

$$\alpha(h) = \lambda (3h + \mu^3)^{1/3}$$

for $\lambda, \mu \in \mathbb{R}$, and the proof is done. ■

Remark 64 *In the notation of the last Lemma, Schwarzschild put the constant $\lambda = 1$ by requiring that*

$$\lim_{h \rightarrow \infty} \frac{[\alpha(h)]^2}{(3h)^{2/3}} = 1$$

since he wanted that his solution in “polar coordinates with determinant 1” approximate the Minkowski spacetime as $h \rightarrow \infty$. In our derivation, we are free to set $\lambda \neq 0$ to whatever we want, since this means only a change in the scale of special coordinates (t, h) . We will, however, stay with the Schwarzschild choice.

Now we can derive the Schwarzschild metric like we did for the Hilbert-Droste case or in Example 52:

Proposition 65 *Let $M = (P, (t, h), f, g, \alpha)$ be a unimodular model. Its spacetime \mathbf{S}_M obeys the Einstein field equation in vacuum and the limit of Remark 64 if and only if there are real numbers k, μ such that*

$$\begin{aligned} \alpha(h) &= (3h + k^3)^{1/3} \\ f(h) &= \frac{[\alpha(h)]^4}{g(h)} = 1 - \frac{\mu}{\alpha(h)} \end{aligned}$$

Proof. The first equation follows from last Lemma. Computing the derivatives of α and using the condition that

$$g(h) = \frac{[\alpha(h)]^4}{f(h)}$$

(recall Definition 61) we get, by the second equation of Proposition 49,

$$f'(h) = \frac{1 - f(h)}{3h + \mu}$$

and the result follows simply by solving this equation. ■

Remark 66 *In the coordinates (t, h) , defined by Definition 61 and Remark 82, the metric described by the last Proposition reads*

$$- \left[1 - \frac{\mu}{\alpha(h)} \right] dt \otimes dt + \frac{1}{[\alpha(h)]^4} \frac{1}{1 - \mu/\alpha(h)} dh \otimes dh + [\alpha(h)]^2 \zeta_{S^2} \quad (1)$$

where $\alpha(h) = (3h + k^3)^{1/3}$ and ζ_{S^2} is the Euclidean metric of S^2 . As α is a diffeomorphism from \mathbb{R}^+ onto \mathbb{R}^+ , we can define (t, R) to be the special coordinates of $P \subset \mathbb{R} \times \mathbb{R}^+$ such that $R = \alpha \circ h$ (recall first paragraph of Section 3.1). Hence, in the (t, R) coordinates, the last metric reads as

$$- \left(1 - \frac{\mu}{R} \right) dt \otimes dt + \frac{1}{1 - \mu/R} dR \otimes dR + R^2 \zeta_{S^2} \quad (2)$$

Thus we have the Schwarzschild metric and we known how to make a transformation such that its coordinate expression is like that of Hilbert-Droste. But we do not have yet the *Schwarzschild solution*. As we did in last section, we must select some submanifold P of $\mathbb{R} \times \mathbb{R}^+$ to fix the topology and the spacetime manifold itself.

Remark 67 *In his original work, Schwarzschild imposed the condition that the metric components must be smooth except in the origin of his coordinate system. However, since our spacetime manifold is $\mathbb{R} \times \mathbb{R}^+ \times S^2$, the only way to realize that condition is by introducing the manifold with boundary $\mathbb{R} \times [0, \infty[\times S^2$ and extending continuously the mappings f, g and α from \mathbb{R}^+ to $[0, \infty[$. Thus, as in the boundary of $\mathbb{R} \times \{0\} \times S^2$ the functions f and g satisfy*

$$\frac{1}{k^4} f(0)g(0) = 1 - \frac{\mu}{k},$$

the Schwarzschild condition is in fact equivalent to

$$k = \mu.$$

So now we are motivated to state the Schwarzschild solution:

Definition 68 (Schwarzschild solution) *Given a real number μ , the Schwarzschild solution $S(\mu)$ is the spacetime \mathbf{S} for which there is an unimodular model $M = (P, (t, h), f, g, \alpha)$ such that $\mathbf{S} = \mathbf{S}_M$,*

$$P = \mathbb{R} \times \mathbb{R}^+$$

$$\alpha(h) = (3h + \mu^3)^{1/3}$$

$$f(h) = \frac{[\alpha(h)]^4}{g(h)} = 1 - \frac{\mu}{\alpha(h)}$$

Problem 69 *Given a real number μ , are the Schwarzschild $S(\mu)$ and Hilbert-Droste $H(\mu)$ solutions equivalents?*

Solution 70 *No, as they have a different topologies. The manifold which describe the Hilbert-Droste solution is*

$$\mathbb{R} \times (\mathbb{R}^+ - \{\mu\}) \times S^2,$$

while that the Schwarzschild manifold is simply

$$\mathbb{R} \times \mathbb{R}^+ \times S^2.$$

Because of its topology, the Hilbert-Droste solution can be sliced into two parts, one called the exterior, whose manifold is $\mathbb{R} \times (\mu, \infty) \times S^2$, and another called the interior (or the *black hole*), whose manifold is $\mathbb{R} \times (0, \mu) \times S^2$. As

we shall see in the next section, this topological property allow the Hilbert-Droste manifold to be glued together with another manifold (recall Section A), constituting what is known by the Kruskal spacetime.

However, since the Schwarzschild manifold is homeomorphic to $\mathbb{R} \times (\mathbb{R} - \{0\}) \times S^2$ (see, for instance, Example 32), we cannot find any manifold to which the Schwarzschild manifold can be glued to, in the sense of Definition 128. Even if we found in Remark 66 a coordinate transformation such that the Schwarzschild metric acquire the same form as the Hilbert-Droste, in the former, the metric expression holds only for $R > \mu$.

Remark 71 *On the other hand, differently from what the author of [20] did, based only on the above discussion, we cannot jump to the conclusion that black holes do not exist as appropriated solutions of Einstein equation. Indeed, the fact that the Einstein field equation have many solutions with black holes seems well established, and in some cases, according to General Relativity, black holes are unavoidable (in gravitational collapses). What is important to keep in mind, however, is that there is no internal mechanism in the theory to decide between the topologies of the Schwarzschild solution and the Hilbert-Droste solution. And, in the last analysis, the existence of black holes or the decision between the above solutions is an experimental quest.*

4 Extending the Hilbert-Droste Solution

In the last Section, two descriptions of the gravitational field of a mass point in General Relativity were discussed, the Schwarzschild and the Hilbert-Droste solutions. However, differently from the first, the manifold of the latter is *not connected*, so it cannot qualify as a legitimate spacetime (cf. Definition 20 and, for a physical motivation, see Remark 21).

In Subsection 4.1, we shall extend the Hilbert-Droste manifold (cf. Definition 35 and Appendix A) in order to obtain a *maximal spacetime*, following a procedure presented by Kruskal in ref. [24] and by Szekeres in ref. [25], both published in 1959. Then, a historical summary (not expected to be complete) of the events which culminated in the approach adopted by Kruskal and Szekeres (namely, the search for new coordinate systems) is presented.

Finally, in Subsection 4.2, we discuss both from a mathematical and a chronological standpoint an alternative extension of the Hilbert-Droste solution by means of an embedding of that solution in a vector manifold, an idea which began in the works of Kasner of 1929 and that was completed by Fronsdal in 1959.

4.1 Kruskal-Szekeres Spacetime

4.1.1 Mathematical Formalism

In what follows, (u, v) will denote the natural coordinates of \mathbb{R}^2 .

For the sake of comparison, we start by defining the Hilbert-Droste plane and spacetime, which are nothing more than a particular case of Definition 44 and a restatement of Definition 57.

Definition 72 *Let μ be a positive real. So, the Hilbert-Droste or simply the HD plane with mass μ is the pseudo-Riemannian manifold $(\mathbb{R} \times (\mathbb{R}^+ - \{\mu\}), \zeta_H)$ such that there exists a coordinate system (t, r) , which will be called the Hilbert-Droste or just HD coordinates, for which*

$$\zeta_H = -\left(1 - \frac{\mu}{r}\right) dt \otimes dt + \frac{dr \otimes dr}{1 - \mu/r}.$$

The Hilbert-Droste or HD black hole and the normal region of the HD plane are the pseudo-Riemannian submanifolds

$$\mathcal{B} = (\mathbb{R} \times (0, \mu), \zeta_H|_{\mathbb{R} \times (0, \mu)})$$

and

$$\mathcal{N} = (\mathbb{R} \times (\mu, \infty), \zeta_H|_{\mathbb{R} \times (\mu, \infty)})$$

respectively.

From now on, for each positive real μ , the HD plane with mass μ will be denote by $\mathcal{Q}_{H(\mu)}$, and the notation of the latter definition will be adopted in what follows. In particular, (t, r) will always denotes HD coordinates.

Definition 73 *Let μ be a positive real. Hence, the Hilbert-Droste or HD solution is the warped product $\mathcal{Q}_{H(\mu)} \times_r S^2$, while its black hole and normal region are, respectively, $\mathcal{B} \times_r S^2$ and $\mathcal{N} \times_r S^2$.*

The reader must keep in mind, however, that the last Definition do *not* define a true spacetime, since it is not connected.

Now we start the construction of the Kruskal-Szekeres spacetime like we did in Section 3.1 for the Schwarzschild case:

Definition 74 *A Kruskal-Szekeres model, or a KS model for brevity, is an ordered list $K = (\mu, P, f, F)$, where μ is a positive real number called the mass of K , P is a submanifold of \mathbb{R}^2 , f is a diffeomorphism from \mathbb{R}^+ onto $[-\mu, \infty[\subset \mathbb{R}$ and F is a smooth mapping from \mathbb{R}^+ into \mathbb{R}^+ .*

Definition 75 *The Kruskal-Szekeres or simply the KS plane associated with a given KS model K is the pseudo-Riemannian manifold (P, ζ_K) such that*

$$\zeta_K = \frac{1}{2} F(r) (du \otimes dv + dv \otimes du)$$

where $r = f^{-1}(uv)$.

On what follows, the KS plane of a KS model K will be denoted by \mathcal{Q}_K , and its metric by ζ_K .

Remark 76 Let $K = (\mu, P, f, F)$ be a KS model. By Problem 17 and by the fact that $\zeta_K(\partial_u - \partial_v, \partial_u - \partial_v) = -F(r) < 0$, the manifold Π_K is time orientable. Then we shall call a vector $X \in T_p(\Pi_K)$ future-pointing if X is in the same causal cone as $\partial_u - \partial_v$.

The next Lemma will be used in the end to prove that the Kruskal-Szekeres spacetime is maximal. The partial derivative $\partial g / \partial x$ of a given mapping g will be denoted below by g_x .

Lemma 77 Let $K = (\mu, P, f, F)$ be a KS model and S the sectional curvature of Π_K . Thus

$$S(u, v) = \frac{2}{F(r)} \left[\frac{F_u(r)}{F(r)} \right]_v.$$

Proof. Let ζ_K and R be the metric and the Riemannian curvature tensor of Π_K , respectively. By Definition (recall the proof of Lemma 50),

$$S = -\frac{\zeta_K(R_{\partial_u \cdot \partial_v} \partial_u, \partial_v)}{\zeta_K(\partial_u, \partial_v)^2}.$$

Let $(\Gamma^k_{ij})_{(k,i,j) \in [1,2]^3}$ be the Christoffel symbols and let $x^1 = u, x^2 = v$. By a direct computation, the only nonzero symbols are

$$\Gamma^1_{11} = \frac{1}{F(r)} F_u(r) \quad \Gamma^2_{22} = \frac{1}{F(r)} F_v(r)$$

Therefore $R_{\partial_u \cdot \partial_v} \partial_u = -D_{\partial_v} \partial_u$ and

$$D_{\partial_v} D_{\partial_u} \partial_u = D_{\partial_v} \left[\frac{1}{F(r)} F_u(r) \partial_u \right] = \left[\frac{F_u(r)}{F(r)} \right]_v \partial_u$$

Finally

$$S = \frac{1}{\zeta_K(\partial_u, \partial_v)} \left[\frac{F_u(r)}{F(r)} \right]_v$$

and the result follows by Definition 75. ■

In the next Definition, we shall divide the KS plane into three (not necessarily connected) submanifolds, a procedure which will be useful in determining in what sense the Kruskal-Szekeres spacetime contains the black hole and the normal region of the HD solution.

Definition 78 Let $K = (\mu, P, f, F)$ be a KS model. So, the Horizon and Regions I and II of Π_K are the pseudo-Riemannian submanifolds

$$\begin{aligned} \mathcal{H}_K &= \{(u, v) \in P : f^{-1}(uv) = \mu\}, \\ \mathcal{R}_I &= \{(u, v) \in P : f^{-1}(uv) \in]0, \mu[\}, \\ \mathcal{R}_{II} &= \{(u, v) \in P : f^{-1}(uv) \in]\mu, \infty[\}, \end{aligned}$$

with the metric inherited from Π_K . The positive and negative parts of \mathcal{R}_I and \mathcal{R}_{II} are respectively

$$\begin{aligned}\mathcal{R}_I^+ &= \{(u, v) \in \mathcal{R}_I : u > 0\}, & \mathcal{R}_I^- &= \{(u, v) \in \mathcal{R}_I : u < 0\}, \\ \mathcal{R}_{II}^+ &= \{(u, v) \in \mathcal{R}_{II} : u > 0\}, & \mathcal{R}_{II}^- &= \{(u, v) \in \mathcal{R}_{II} : u < 0\}.\end{aligned}$$

We shall adopt the notation of the last Definition for the rest of this section.

The following Lemma will be our main connection between the KS plane and the HD solution.

Lemma 79 *Let $K = (\mu, P, f, F)$ be a KS model and \mathcal{B} and \mathcal{N} the black hole and the normal region of $\mathcal{Q}_{H(\mu)}$, respectively. So \mathcal{R}_I^+ and \mathcal{R}_I^- are isometric to \mathcal{B} while \mathcal{R}_{II}^+ and \mathcal{R}_{II}^- are isometric to \mathcal{N} if*

$$\begin{aligned}f(r) &= (r - \mu) \exp\left(\frac{r}{\mu}\right), \\ F(r) &= \frac{4\mu^2}{r} \exp\left(-\frac{r}{\mu}\right).\end{aligned}$$

Proof. Let $\xi : \mathcal{R}_{II}^+ \rightarrow \mathcal{N}$ be the mapping such that

$$u \circ \xi(t, r) = \sqrt{|r - \mu|} \exp\left(\frac{r + t}{2\mu}\right), \quad (3)$$

$$v \circ \xi(t, r) = \sqrt{|r - \mu|} \exp\left(\frac{r - t}{2\mu}\right), \quad (4)$$

are surjections $\mathbb{R} \times]\mu, \infty[\rightarrow \mathbb{R}^+$. Hence, ξ is a diffeomorphism.

As $du = (u \circ \xi)_t dt + (u \circ \xi)_r dr$ and $dv = (v \circ \xi)_t dt + (v \circ \xi)_r dr$, by Definition 75,

$$\begin{aligned}\zeta_K &= \frac{1}{2} F(r) (du \otimes dv + dv \otimes du) = \\ &F(r) (u \circ \xi)_t (v \circ \xi)_t dt \otimes dt + F(r) (u \circ \xi)_r (v \circ \xi)_r dr \otimes dr + \\ &\frac{1}{2} F(r) [(u \circ \xi)_t (v \circ \xi)_r + (u \circ \xi)_r (v \circ \xi)_t] (dt \otimes dr + dr \otimes dt).\end{aligned}$$

Then, by Definition 72, we have an isometry if and only if

$$F(r) (u \circ \xi)_t (v \circ \xi)_t = -\left(1 - \frac{\mu}{r}\right), \quad (5)$$

$$F(r) (u \circ \xi)_r (v \circ \xi)_r = \frac{1}{1 - \mu/r}, \quad (6)$$

$$(u \circ \xi)_t (v \circ \xi)_r + (u \circ \xi)_r (v \circ \xi)_t = 0. \quad (7)$$

By computing the derivatives, Eq.(7) holds trivially and Eqs.(5) and (6) are satisfied only if we choose $F(r)$ as in the Proposition. Finally,

$$(u \circ \xi)(v \circ \xi) = (r - \mu) \exp\left(\frac{r}{\mu}\right)$$

So $r = f^{-1}(uv)$ if $f(r) = (r - \mu) \exp(r/\mu)$.

The same reasoning holds for \mathcal{R}_{II}^- if we redefine the diffeomorphism $\xi : \mathcal{R}_{II}^- \rightarrow \mathcal{N}$ such that

$$u \circ \xi(t, r) = -\sqrt{|r - \mu|} \exp\left(\frac{r + t}{2\mu}\right), \quad (8)$$

$$v \circ \xi(t, r) = -\sqrt{|r - \mu|} \exp\left(\frac{r - t}{2\mu}\right), \quad (9)$$

Finally, to prove the isometry between \mathcal{R}_I^+ and \mathcal{R}_I^- and \mathcal{B} , just change the domains and the signs of $u \circ \xi$ and $v \circ \xi$. ■

Remark 80 *The reader may inquire how did we found the equations for $u \circ \xi$ and $v \circ \xi$ in the proof of the last Proposition. Indeed, an algebraic manipulation of Eqs. (5), (6) and (7) gives*

$$\begin{aligned} |(u \circ \xi)_t| &= \left(1 - \frac{\mu}{r}\right) (u \circ \xi)_r, \\ |(v \circ \xi)_t| &= \left(1 - \frac{\mu}{r}\right) (v \circ \xi)_r. \end{aligned}$$

Then, after choosing a sign for the left-hand side (cf. the following Remark), simply use separation of variables and apply some obvious contour conditions. The details are left as an easy exercise.

Remark 81 *We can interchange the signals of the t -coordinate in Eq.(3) and in Eq.(4) without affecting the proof of the latter Lemma. However, our particular choice is the only one in which our time-orientation of Remark 76 is consistent with ∂_t being future-pointing, since*

$$\frac{\partial}{\partial t} = \frac{\sqrt{|r - \mu|}}{2\mu} \left[\exp\left(\frac{r + t}{2\mu}\right) \frac{\partial}{\partial u} - \exp\left(\frac{r - t}{2\mu}\right) \frac{\partial}{\partial v} \right]$$

imply that

$$g(\partial_t, \partial_u - \partial_v) = -\frac{\sqrt{|r - \mu|}}{2\mu} \left[\exp\left(\frac{r + t}{2\mu}\right) + \exp\left(\frac{r - t}{2\mu}\right) \right] g(\partial_u, \partial_v) < 0.$$

In the following Exercise, the reader is invited to prove why the KS space-time (cf. the Definition 86 below) have what some writers call a “fundamental singularity” at “ $r = 0$ ”.

Exercise 82 *Let $K = (\mu, P, f, F)$ be a KS model with f and F given by Lemma 79, S the sectional curvature of \mathcal{Q}_K and define $r = f^{-1}(uv)$. Using Lemma 77, prove by a direct computation that $\lim_{r \rightarrow 0} S = \infty$. Hint: use the following facts: $\lim_{r \rightarrow 0} u, \lim_{r \rightarrow 0} v \neq 0$, $F_u(r) = F'(r)r_u$,*

$$\left[\frac{F_u(r)}{F(r)} \right]_v = \left[\frac{F'(r)r_u}{F(r)} \right]_v = \frac{F'(r)}{F(r)} r_{uv} + \left[\frac{F'(r)}{F(r)} \right]_v r_u$$

and calculate r_u, r_{uv} implicitly by $f(r) = uv$ in Lemma 79.

Therefore, because the sectional curvature in a 2-dimensional manifold as the KS plane becomes a real-valued mapping $\mathcal{Q}_K \rightarrow \mathbb{R}$, depending exclusively of the pseudo-Riemannian structure of \mathcal{Q}_K without any mention to a coordinate system, the “singularity” expressed in the limit $\lim_{r \rightarrow 0} S = \infty$ means that the pseudo-Riemannian structure of \mathcal{Q}_K itself cannot be defined in the region where $f^{-1}(uv) = r = 0$, justifying the name “fundamental singularity”.

Now, after the following list of Definitions and Remarks, the Kruskal-Szekeres spacetime will be finally defined.

Definition 83 (KS spacetime plane) *Let $K = (\mu, P, f, F)$ be a KS model with*

$$P = \{(u, v) \in \mathbb{R} \times \mathbb{R} : uv > -\mu\}$$

and f and F as defined in Lemma 79. In this case, the KS plane \mathcal{Q}_K is called a spacetime plane with mass μ .

Hereinafter, a KS spacetime plane with mass μ will be denoted by $\mathcal{Q}_{K(\mu)}$, in analogy with the HD plane $\mathcal{Q}_{H(\mu)}$.

Remark 84 *The choice for manifold P in Definition 83 is based on the fact that the image of \mathbb{R}^+ under f is $[-\mu, \infty[$, so that $-\mu < f(r) = uv < \infty$.*

Remark 85 *By Lemma 79, there is an isometry ξ from \mathcal{R}_I^+ (respect. \mathcal{R}_I^-) onto the HD black hole \mathcal{B} and another isometry, say, η , from \mathcal{R}_{II}^+ (respect. \mathcal{R}_{II}^-) onto the HD normal region \mathcal{N} . Denoting again by (t, r) HD coordinates, $(t, r) \circ \xi$ and $(t, r) \circ \eta$ are charts of \mathcal{R}_I^+ (respect. \mathcal{R}_I^-) and \mathcal{R}_{II}^+ (respect. \mathcal{R}_{II}^-). Together, these mappings establish a coordinate system for the manifold union $\mathcal{R}_I^+ \cup \mathcal{R}_{II}^+$ (respect. $\mathcal{R}_I^- \cup \mathcal{R}_{II}^-$), and will be called Hilbert-Droste or HD coordinates on the KS spacetime plane.*

Definition 86 (KS spacetime) *Let μ be a positive real number and f as in Lemma 79. The Kruskal-Szekeres spacetime, or KS spacetime for brevity, with mass μ is the warped product $\mathcal{Q}_{K(\mu)} \times_r S^2$ where $r = f^{-1}(uv)$, and its Horizon is the submanifold such that $uv = f(\mu) = 0$.*

The geometric properties of the Horizon, which will be used in the next Subsection to compare the KS spacetime with the Fronsdal embedding of the Hilbert-Droste solution, are summarized in

Lemma 87 *The Horizon \mathcal{H} of the KS spacetime with mass μ is mapped onto S^2 by a homothety of coefficient μ .*

Proof. Since $uv = 0$ at the Horizon, then $d(u|\mathcal{H}) = 0$ or $d(v|\mathcal{H}) = 0$. In any case, the metric of $\mathcal{Q}_{K(\mu)}$ degenerates (cf. Definition 75) and the metric of the KS spacetime $\mathcal{Q}_{K(\mu)} \times_r S^2$ becomes just $\mu^2 \zeta_{S^2}$, where ζ_{S^2} is the Euclidean metric of S^2 . ■

Remark 88 *The above Lemma is pictured as saying that the Horizon is a 2-dimensional sphere with radius μ surrounding the “fundamental singularity” at $r = 0$. That is the reason why the traditional literature calls $r = \mu$ the “Schwarzschild radius”.*

Our first step to prove that the KS spacetime is the maximal extension of the HD solution is to extend Lemma 79 from the KS and HD planes to the KS and HD “spacetimes”.

In the proof of the next Lemma, we shall employ the natural projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ given by $\pi_1(p, q) = p$ and $\pi_2(p, q) = q$, for any not empty sets X and Y .

Lemma 89 *Let f be the mapping given by Definition 79 and define again $r = f^{-1}(uv)$. Let \mathcal{B} and \mathcal{N} be the black hole and normal region of $\mathcal{Q}_{H(\mu)}$ and \mathcal{R}_I^+ , \mathcal{R}_I^- , \mathcal{R}_{II}^+ , \mathcal{R}_{II}^- the submanifolds of $\mathcal{Q}_{K(\mu)}$ as in Definition 78. So $\mathcal{R}_I^+ \times_r S^2$ and $\mathcal{R}_I^- \times_r S^2$ are isometric to $\mathcal{B} \times_r S^2$ while $\mathcal{R}_{II}^+ \times_r S^2$ and $\mathcal{R}_{II}^- \times_r S^2$ are isometric to $\mathcal{N} \times_r S^2$.*

Proof. Let $g, h, \zeta_I, \zeta_{\mathcal{N}}$ and ζ_{S^2} be the metric tensors of $\mathcal{R}_I^+ \times_r S^2$, $\mathcal{N} \times_r S^2$, \mathcal{R}_I^+ , \mathcal{N} and S^2 . Letting $\xi : \mathcal{R}_I^+ \rightarrow \mathcal{N}$ be the isometry whose existence was proved in Lemma 79, define $\eta : \mathcal{R}_I^+ \times_r S^2 \rightarrow \mathcal{N} \times_r S^2$ by $\eta(p, q) = (\xi(p), q)$. Let

$$(v, w) \in T_{(p,q)}(\mathcal{R}_I^+ \times_r S^2) \times T_{(p,q)}(\mathcal{R}_I^+ \times_r S^2)$$

be tangent vectors at (p, q) in $\mathcal{R}_I^+ \times_r S^2$, so that $v = \pi_{1*}v + \pi_{2*}v$ for $(\pi_{1*}v, \pi_{2*}v) \in T_p(\mathcal{R}_I^+) \times T_q(S^2)$ and the same for w (recall Lemma 23 for a justification of the latter notation). So,

$$\begin{aligned} \eta^*h(v, w) &= \zeta_{\mathcal{N}}(\pi_{1*} \circ \eta_*, \pi_{1*} \circ \eta_*v) + r^2 \zeta_{S^2}(\pi_{2*} \circ \eta_*u, \pi_{2*} \circ \eta_*v) \\ &= \zeta_{\mathcal{N}}(\xi_* \circ \pi_{1*}u, \xi_* \circ \pi_{1*}v) + r^2 \zeta_{S^2}(\pi_{2*}u, \pi_{2*}v) \\ &= \zeta_I(\pi_{1*}u, \pi_{1*}v) + r^2 \zeta_{S^2}(\pi_{2*}u, \pi_{2*}v) \\ &= g(v, w) \end{aligned}$$

Hence, $\eta^*h = g$. The same argument can be repeated in order to demonstrate the others isometries. ■

Finally, we start our procedure to prove that the Kruskal-Szekeres manifold is indeed maximal

On what follows, all geodesics γ defined on some $I \subset \mathbb{R}$ are *future-pointing*, in the sense that $\gamma'(t) \in \sec T_{\gamma(t)}M$ is future-pointing for all $t \in I$.

Lemma 90 *Let $\gamma : I \subset \mathbb{R} \rightarrow \mathcal{Q}_{K(\mu)}$ be an inextendible null geodesic. Then there exists some $\varepsilon \in \{-1, +1\}$, a diffeomorphism ϕ from $J = \{s \in \mathbb{R} : \varepsilon s > 0\}$ onto some subset of I and HD coordinates (t, r) on $\mathcal{Q}_{K(\mu)}$ such that*

$$\begin{aligned} r \circ \gamma \circ \phi(s) &= \varepsilon s, \\ t \circ \gamma \circ \phi(s) &= s + \varepsilon \mu \log |\mu - \varepsilon s|, \end{aligned}$$

for all $\varepsilon s \in J - \{\mu\}$.

Proof. Let $\gamma_0 = t \circ \gamma$ and $\gamma_1 = r \circ \gamma$ be so that $\gamma'(t) = \gamma'_0(t)\partial_t + \gamma'_1(t)\partial_r$, where $'$ denotes ordinary derivative and $\{\partial_t, \partial_r\}$ is the set of coordinate vector fields. By the properties of the Levi-Civita connection and the fact that ∂_t is a Killing vector field, we have

$$\begin{aligned}\zeta_K(\gamma'(t), \gamma'(t)) &= 0, \\ \zeta_K(\gamma'(t), \partial_t|_{\gamma t}) &= -K,\end{aligned}$$

for $K > 0$ since that $\gamma'(t)$ and $\partial_t|_{\gamma t}$ are in the same causal cone. Then

$$\begin{aligned}\gamma'_1(t) &= K \text{ or } \gamma'_1(t) = -K \\ \gamma'_0(t) &= \frac{J}{1 - \mu/\gamma_1(t)}\end{aligned}$$

So there is a unique $\varepsilon \in \{-1, +1\}$ such that $\gamma'_1(t) = \varepsilon K$. Hence $\gamma_1(t) = \varepsilon(Kt + C)$ for $C \in \mathbb{R}$. But γ is inextendible, so $\phi(s) = (s - C)/K$ must be a diffeomorphism from $J = \{s \in \mathbb{R} : \varepsilon s > 0\}$ onto some subset of I (if not, $(\gamma \circ \phi(J)) \cap I \neq \emptyset$ and γ would not be inextendible). Thus $\gamma_1 \circ \phi(s) = \varepsilon s$ and

$$(\gamma_2 \circ \phi)'(s) = \frac{1}{1 - \mu/(\varepsilon s)}$$

Hence, there is $D \in \mathbb{R}$ for which $\gamma_2 \circ \phi(s) = D + s + \varepsilon \mu \log|\mu - \varepsilon s|$, and result follows by the fact that $(t, r) \rightarrow (t + D, r)$ is an isometry. ■

Finally:

Proposition 91 *The KS spacetime is maximal.*

Proof. As S^2 is connected and compact, we just need to prove that the KS spacetime plane $\mathcal{Q}_{K(\mu)}$ is maximal. Let f be as in Lemma 79 and, as usual, define $r = f^{-1}(uv)$. Let M be a spacetime in which $\mathcal{Q}_{K(\mu)}$ is a submanifold, γ an inextendible null geodesic in M and $J \subset I$ the largest subset such that $\gamma(J) \subset \mathcal{Q}_{K(\mu)}$. Assume that J is not empty. By Lemma 90, as γ is inextendible, $k \in I - J$ implies $r \circ \gamma(k) = 0$ or $r \circ \gamma(k) = \mu$. But by Exercise 82, in the first case: $\lim_{t \rightarrow k} S \circ \gamma(t) = \infty$. So $r \circ \gamma(k) = \mu$. However, this imply $\gamma(k) \in H \subset \mathcal{Q}_{K(\mu)}$. So $I - J = \emptyset$ and the result follows by Lemma 41. ■

In the following Problem, the reader is invited to see why the “interior submanifold” \mathcal{R}_I^+ of the HD solution is called a *black hole*, while \mathcal{R}_I^- is usually know as a *white hole*.

Problem 92 *Let $\gamma : I \subset \mathbb{R} \rightarrow \mathcal{Q}_{K(\mu)}$ be a (future-pointing) null geodesic. Suppose there exists some $k \in I$ such that $\gamma(k) \in \mathcal{R}_I^+$ (respect. \mathcal{R}_I^-). Hence $\gamma|(I \cap [k, \infty)) \subset \mathcal{R}_I^+$ (respect. $\exists s \in I : \gamma|(I \cap [s, \infty)) \subset \mathcal{R}_I^-$).*

Solution 93 In \mathcal{R}_I^+ , dr is a timelike vector, and so is $\text{grad } r$. Hence, as $v < 0$ and $u > 0$ in \mathcal{R}_I^+ ,

$$\zeta_K(\text{grad } r, \partial_u - \partial_v) = \frac{\partial r}{\partial u} - \frac{\partial r}{\partial v} = \frac{v - u}{r} \exp\left(\frac{r}{\mu}\right) < 0$$

So $\text{grad } r$ is future-pointing. Therefore

$$(r \circ \gamma)'(t) = dr[\gamma'(t)] = \zeta_K(\text{grad } r, \gamma'(t)) < 0$$

Now, in \mathcal{R}_I^- , $v > 0$ and $u < 0$ so that $\zeta_K(\text{grad } r, \partial_u - \partial_v) > 0$. Therefore, $\text{grad } r$ is past-pointing in \mathcal{R}_I^- , so that $(r \circ \gamma)'(t) > 0$.

At this point, the reader may consult the Appendix B to see our brief discussion concerning some exotic topological objects associated with the KS spacetime. There, we shall prove the important fact that there exists no geodesic which starts on the region \mathcal{R}_{II}^+ and goes to \mathcal{R}_{II}^- and vice-versa, or that starts on \mathcal{R}_I^+ and goes to \mathcal{R}_I^- .

This result has been pictured by some writers as saying that \mathcal{R}_{II}^+ and \mathcal{R}_{II}^- (\mathcal{R}_I^+ and \mathcal{R}_I^- , respectively) belongs to distinct “universes” which are connected by the horizon \mathcal{H} of the KS spacetime. Some authors, e.g., Kruskal himself [24], have compared the latter object to a “wormhole”, in the Misner and Wheeler sense, or to a “bridge”, in the Einstein-Rosen sense. However, because one cannot travel through it without falling in the “fundamental singularity”, the more accurate term *horizon* has been adopted in the literature.

Because it is impossible even in principle to verify the existence of the another “universe” represented by a manifold union like $\mathcal{R}_I^- \cup \mathcal{R}_{II}^-$, we may ignore it and *truncate* the KS spacetime by removing $\mathcal{R}_I^- \cup \mathcal{R}_{II}^-$ from the KS spacetime manifold. This is done in many textbooks, cf. for instance ref. [12] and ref. [14]. However, it is important to keep in mind that the *maximal* manifold is the KS spacetime *with the two “universes”*, so that, for our geometrical purposes, we shall let Definition 86 as it is without truncating the spacetime.

4.1.2 Historical Overview: Hunting for New Coordinates

The maximal extension of the Hilbert-Droste solution is an important chapter in the history of the General Relativity and could very well deserve an entire work dedicated to it. In the next pages, we give a brief summary of the research programme established from the decade of 1920 to the end of 1960, which is completed in the following Section with a detailed discussion of the Fronsdal embedding.

In this section, let Q_K be the Kruskal-Szekeres spacetime plane and denote by (t, r) the Hilbert-Droste chart on Q_K .

The idea of finding a coordinate system which could remove the singularity of the Hilbert-Droste metric at $r = \mu$ was pioneered by the French mathematician and former Prime Minister of Third Republic P. Painlevé, in 1921 [28]. A year later, the same coordinates proposed by Painlevé were restated by the Swedish

ophthalmologist A. Gullstrand [29], who won the *Nobel Prize in Physiology or Medicine* for his works on the eye optics, using methods of applied mathematics [30].

They introduced a new coordinate, say, z , for which the metric of Q_K could be rewritten as

$$\zeta_K = - \left(1 - \frac{\mu}{r}\right) dz \otimes dz + (dz \otimes dr + dr \otimes dz)$$

Using the same arguments of Lemma 79 and Remark 80, one can find the transformation identity of z in terms of the Hilbert-Droste coordinates, which gives

$$z = t + r + \mu \log \left| \frac{r}{\mu} - 1 \right|.$$

For more details on this coordinates, see [31].

Remark 94 *Ironically, Gullstrand was the member of the Nobel Committee for Physics who argue against Einstein receiving the Nobel Prize for his works in the Relativity Theory. Consult, for instance, [32].*

The spokesman for Relativity Theory in the English-speaking world during the first great war, A. Eddington, found another coordinate system in 1924 which also removed the $r = \mu$ singularity [33]. He introduced a new “time” coordinate \tilde{t} such that

$$\zeta_K = - \left(1 - \frac{\mu}{r}\right) d\tilde{t} \otimes d\tilde{t} + \frac{\mu}{r} (d\tilde{t} \otimes dr + dr \otimes d\tilde{t}) + \left(1 + \frac{\mu}{r}\right) dr \otimes dr$$

and, as in Remark 80, one can find that

$$\tilde{t} = t - \mu \log |r - \mu|.$$

The (\tilde{t}, r) coordinates were finally used by D. Finkelstein in 1958 [34] in order to find the maximal extension of the Hilbert-Droste “spacetime”. However, in the decade of 1920, Eddington’s motivation was very different from Finkelstein’s. The former was concerned with the fact that the Whitehead gravitational theory gave the same predictions as the Relativity Theory for the solar system. Eddington, using the last coordinate expression for the metric, prove that, indeed, the Hilbert-Droste metric was a solution of both gravitational theories.

Another step was given in 1933 by the proposer of the expansion of the universe and of the “primeval atom” theory, G. Lemaître. He introduced a set of coordinates (τ, ρ) for which

$$\begin{aligned} \zeta_K &= -d\tau \otimes d\tau + \frac{\mu}{r} d\rho \otimes d\rho, \\ r &= \left[\frac{2}{3} \sqrt{\mu} (\rho - \tau) \right]^{2/3} \end{aligned}$$

and, as he wrote in his paper [35],

La singularité du champ de Schwarzschild est donc une singularité fictive, analogue à celle qui se présentait à l’horizon du centre dans la forme originale de l’univers de De Sitter.

Remark 95 *Some people argues that C. Lanczos was in fact the first to institute the idea of a “fictitious” singularity. In opposition to Lemaître, he did this by introducing a singularity in a solution which were known as regular. See [36].*

The maximal extension of the Hilbert-Droste “spacetime” became a theme of mainstream research more or less in 1949, with the publication of a *Letter to the Editor* in the *Nature* magazine [37] by J. Synge, an Irish mathematician and physicist who made contributions to differential geometry (Synge’s theorem) and to theoretical physics. He summarized the state of affairs as follows:

“The usual exterior Schwarzschild line element shows an obvious singularity for a certain value of r , say $r = a$. Since a is in every known case much smaller than the radius of the spherical body producing the field, the existence of the singularity appears to be of little interest to astronomers. At the other end of the scale, a discussion of the gravitational field of an ultimate particle, without reference to electromagnetism or quantum theory, might appear equally devoid of physical meaning.

Nevertheless, it does not seem right to leave the theory of the gravitational field of a particle (I mean a point-mass) uncompleted merely because there is no direct physical application. The existence of the singularity at $r = a$ is strange and demands investigation. Investigation shows that there is no singularity at $r = a$; the apparent singularity in the Schwarzschild line element is due to the coordinates employed and may be removed by a transformation, so that there remains no singularity except at $r = 0$ ”.

A year after, Synge published a paper [38] where he also proposed some new coordinates (u, v) for which the metric could be given by

$$\zeta_K = - (1 + v^2 G) du \otimes du + \frac{1}{2} uv (du \otimes dv + dv \otimes du) + (1 - u^2 G) dv \otimes dv,$$

with

$$G = \frac{1}{u^2 - v^2} \left(1 - \frac{4\mu^2 \tanh^2 \xi}{u^2 - v^2} \right) \text{ if } u^2 - v^2 \geq 0,$$

$$G = \frac{1}{u^2 - v^2} \left(1 + \frac{4\mu^2 \tanh^2 \eta}{u^2 - v^2} \right) \text{ if } u^2 - v^2 < 0,$$

where

$$r = \mu \cosh^2 \xi \text{ if } u^2 - v^2 \geq 0,$$

$$r = \mu \cos^2 \eta \text{ if } u^2 - v^2 < 0.$$

Therefore, he preceded to study the motion of geodesics falling into the “interior region $r < \mu$ ” and became the first one to explore the physics of the Hilbert-Droste black hole.

Remark 96 *It is notable that Synge used a coordinate system much more complicated than that used by Eddington and Finkelstein or even the one used by Lemaître (which he knew very well). However, it is not certain if Synge was aware of the coordinates employed by Painlevé and Gullstrand which, even being the first one to appear, are relatively simpler. In fact, there are some authors defending that the latter coordinates are better employed even in a pedagogical context. See [31].*

After the endeavour of Synge and then by Finkelstein, the American mathematical physicist M. Kruskal and the Hungarian-Australian mathematician G. Szekeres employed a comparatively simpler coordinate system to remove the singularity at $r = \mu$. They required that the coordinates (x, y) are to be such that

$$\zeta_K = -F(r)dx \otimes dx + F(r)dy \otimes dy.$$

The reader may then take as an Exercise to prove, using the techniques of Lemma 79, that

$$\begin{aligned} [F(r)]^2 &= \frac{16\mu}{r} \exp\left(-\frac{r}{\mu}\right), \\ r &= f^{-1}(x^2 - y^2), \end{aligned}$$

where

$$f(r) = \left[\left(\frac{r}{\mu} \right) - 1 \right] \exp\left(\frac{r}{\mu}\right)$$

and the coordinate transformations are given by

$$x = \sqrt{\left(\frac{r}{\mu}\right) - 1} \exp\left(\frac{r}{2\mu}\right) \cosh\left(\frac{t}{2\mu}\right), \quad (10)$$

$$y = \sqrt{\left(\frac{r}{\mu}\right) - 1} \exp\left(\frac{r}{2\mu}\right) \sinh\left(\frac{t}{2\mu}\right). \quad (11)$$

The resemblance between these coordinates and that used in the last section (recall, for instance, Lemma 79) is not a coincidence. The chart used in Section 4.1 is just a reformulation of that used by Kruskal and Szekeres, and appeared already in the first edition of [15].

4.2 Kasner-Fronsdal Embedding

There is another aspect of the history of the maximal extension which started already in 1921 with the American mathematician E. Kasner, pupil of F. Klein and D. Hilbert. Kasner is celebrated by his works in Differential Geometry,

General Relativity and even the popularization of the term *googol* to refer to the number 10^{100} , which would later inspire the name of the famous Internet search engine.

Kasner started by proving in [39] that it is impossible to find an imbedding in a flat 5-dimensional manifold of a (non-Euclidean) spacetime which satisfies Einstein field equation in vacuum. (His proof is remarkably simple and elementary). And in a following paper [40] (published with the former in the same Volume of the American Journal of Mathematics), Kasner demonstrated how the Schwarzschild solution can be “embedded” (with a topological defect, cf. the paragraph following Lemma 101) in a flat 6-dimensional manifold.

His latter work will be described in details below.

Remark 97 In [41], C. Fronsdal observed correctly that the embedding proposed by Kasner was not valid for the whole Hilbert-Droste “spacetime”. Fronsdal argues that he imbedded only the “exterior solution $r > \mu$ ”, with an additional topological modification.

First, we define the manifold in which we pretend to imbed the Schwarzschild spacetime:

Definition 98 The Kasner manifold is the 6-dimensional pseudo-Riemannian vector manifold V for which there is a natural isomorphism (u_1, u_2, \dots, u_6) from V onto \mathbb{R}^6 , which will be called the natural coordinates of V , such that the metric ζ_V can be written as

$$\zeta_V = - \sum_{i \in [1,2]} du_i \otimes du_i + \sum_{i \in [3,6]} du_i \otimes du_i$$

In order to simplify his work, Kasner introduced a new coordinate system in the Schwarzschild spacetime. On what follows, the Euclidean metric of $\mathbb{R}^3 - \{0\}$ will be denoted by $\zeta_{\mathbb{R}^3}$.

Definition 99 Let $S = S(\mu)$ be a Schwarzschild solution (cf. Definition 68) and (t, R) a chart of the Schwarzschild plane Π of S such that the metric ζ_S of S is given by

$$\zeta_S = - \left(1 - \frac{\mu}{R}\right) dt \otimes dt + \frac{1}{1 - \mu/R} dR \otimes dR + \zeta_{\mathbb{R}^3}$$

for $R > \mu$ (cf. Remarks 66 and 67). Let $h \rightarrow \phi(h) = \sqrt{4\mu^2(-1 + h/\mu)}$ be a diffeomorphism from $] \mu, \infty[\subset \mathbb{R}$ onto \mathbb{R}^+ . So, the Kasner chart is the coordinate system (t, H) on Π for which $H = \phi \circ R$.

Exercise 100 Assuming the notation of the last Definition, prove that the metric ζ_S of S is, in the Kasner chart, given by

$$\zeta_S = - \frac{H^2}{H^2 + 4\mu^2} dt \otimes dt - dH \otimes dH + \zeta_{\mathbb{R}^3}.$$

The next Lemma will finally give the imbedding condition. As usual, the partial derivative $\partial g/\partial x$ of a given mapping g will be denoted by g_x .

Lemma 101 *Let V and $S = S(\mu)$ be the Kasner manifold and the Schwarzschild solution with mass μ , and denote by ζ_V and ζ_S their respective metrics. Let (u_1, u_2, \dots, u_6) be the natural coordinates of V , (t, H) the Kasner chart of the Schwarzschild plane of S and (t, x) a coordinate system of S related to (t, H) by*

$$H = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad x = (x_1, x_2, x_3).$$

So, a mapping ξ from S onto some submanifold $\mathbb{k} \subset V$ is an isometry if

$$\begin{aligned} u_1 \circ \xi(t, x) &= \frac{H}{\sqrt{H^2 + 4\mu^2}} \sin t, \\ u_2 \circ \xi(t, x) &= \frac{H}{\sqrt{H^2 + 4\mu^2}} \cos t, \\ u_3 \circ \xi(t, x) &= \int_0^H \sqrt{1 + \frac{16\mu^4}{(H^2 + 4\mu^2)^3}} dH \end{aligned}$$

and

$$u_4 \circ \xi(t, x) = x_1, \quad u_5 \circ \xi(t, x) = x_2, \quad u_6 \circ \xi(t, x) = x_3.$$

In this case, \mathbb{k} will be called the Kasner imbedding.

Proof. Applying the condition $\xi^*(\zeta_V|_{\mathbb{k}}) = \zeta_S$ to the coordinate vectors ∂_t, ∂_H , we derive that

$$\begin{aligned} (u_1 \circ \xi)_r^2 + (u_2 \circ \xi)_r^2 - (u_3 \circ \xi)_r^2 &= 1, \\ (u_1 \circ \xi)_t^2 + (u_2 \circ \xi)_t^2 - (u_3 \circ \xi)_t^2 &= \frac{H^2}{H^2 + 4\mu^2}, \\ (u_1 \circ \xi)_t(u_1 \circ \xi)_r + (u_2 \circ \xi)_t(u_2 \circ \xi)_r - (u_3 \circ \xi)_t(u_3 \circ \xi)_r &= 0. \end{aligned}$$

We set $u_4 \circ \xi(t, x) = x_1$, $u_5 \circ \xi(t, x) = x_2$, $u_6 \circ \xi(t, x) = x_3$. Now, let F, W be mappings from \mathbb{R}^+ into \mathbb{R} and let G, Z be mappings from \mathbb{R} into \mathbb{R} , such that $u_1 \circ \xi(t, x) = G(t)F(H)$, $u_2 \circ \xi(t, x) = Z(t)F(H)$ and $u_3 \circ \xi(t, x) = W(H)$. Then from the last equation,

$$[G(t)^2 + Z(t)^2]' = 0,$$

which holds if we choose $G(t) = \sin t$ and $Z(t) = \cos t$. But by the second,

$$[F(H)]^2 [G'(t)^2 + Z'(t)^2] = \frac{H^2}{H^2 + 4\mu^2},$$

hence $F(H) = H/\sqrt{H^2 + 4\mu^2}$. Finally,

$$[F'(H)]^2 = 1 + [W'(H)]^2$$

follows from the first equation. ■

Assuming the notation of the last Lemma, \mathbb{k} is then an imbedding of the Schwarzschild spacetime into the Kasner manifold. However, the points (t, x) and $(t + 2\pi, x)$ are identified in \mathbb{k} , and therefore, such an imbedding have the exotic topology of $S^1 \times \mathbb{R}^+ \times S^2$, providing an example of what we may call a *naive time machine*.

Remark 102 *Another naive time machine can be “constructed” as follows. Let \mathcal{M} be the Minkowski spacetime and let \approx be the equivalence relation in \mathcal{M} such that $(t, x, y, z) \approx (t + 2\pi, x, y, z)$ in a Lorentz system. So \mathcal{M}/\approx , homeomorphic to $S^1 \times \mathbb{R}^3$, is the simpler case of a naive time machine!*

Remark 103 *We have again an illustration of the topological arbitrariness that exists in General Relativity. In the present case, the choice between the Schwarzschild manifold with the topology of $\mathbb{R} \times \mathbb{R}^+ \times S^2$, and not with that of $S^1 \times \mathbb{R}^+ \times S^2$, is a matter of experimentation: we know that there is no such a time machine in our solar system. (Compare this with the situation discussed in Remark 71).*

As we would expect from our earlier discussion, the Kasner imbedding do not provide a completion for the Schwarzschild solution, since the diffeomorphism $]\mu, \infty[\xrightarrow{\phi} \mathbb{R}^+$ of Definition 99 cannot be extended to $]0, \mu[$.

Lastly, in order to imbed the Hilbert-Droste disconnected “spacetime”, the American theoretical physicist C. Fronsdal completed the maximal extension programme in 1959 by modifying the Kasner manifold. Historically, one of his motivations was to remove the *naive time machine*. As he wrote in [41] (where $Z_1 = u_1 \circ \xi$ and $Z_2 = u_2 \circ \xi$)

Another shortcoming (...) is that Z_1 and Z_2 are periodic functions of t , so that the embedding identifies distinct points of the original manifold. This suggests replacing the trigonometric functions by hyperbolic functions.

To make a parallel with the Kasner work, we present the Fronsdal construction by modifying the Definition 98.

Definition 104 *The Fronsdal manifold is the 6-dimensional pseudo-Riemannian vector manifold U for which there is a natural isomorphism (u_1, u_2, \dots, u_6) from U onto \mathbb{R}^6 , which will be called the natural coordinates of U , such that the metric ζ_U can be written as*

$$\zeta_U = -du_1 \otimes du_1 + \sum_{i \in [2, 6]} du_i \otimes du_i.$$

As we shall see in the proof of the next Lemma, the change of sign from the metric of Definition 98 to the above is necessary because of the hyperbolic mappings adopted Fronsdal.

Lemma 105 *Let U and $H = H(\mu)$ be the Fronsdal manifold and the Hilbert-Droste solution with mass μ , and denote by ζ_U and ζ_H their respective metrics. Let $\mathcal{B} \times_r S^2$ and $\mathcal{N} \times_r S^2$ be the black hole and the normal region of H , (u_1, u_2, \dots, u_6) the natural coordinates of U , (t, r) the Hilbert-Droste coordinates of the plane of H and (t, x) the coordinate system of H related to (t, r) by*

$$r = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad x = (x_1, x_2, x_3).$$

So, the mappings $B \xrightarrow{\eta} \mathbb{K}_B$ and $N \xrightarrow{\xi} \mathbb{K}_N$ where $\mathbb{K}_B, \mathbb{K}_N \subset U$ are submanifolds are isometries if

$$\begin{aligned} u_1 \circ \xi(t, x) &= 2\mu \sqrt{1 - \frac{\mu}{r}} \sinh\left(\frac{t}{2\mu}\right), \quad u_1 \circ \eta(t, x) = 2\mu \sqrt{\frac{\mu}{r} - 1} \cosh\left(\frac{t}{2\mu}\right), \\ u_2 \circ \xi(t, x) &= 2\mu \sqrt{1 - \frac{\mu}{r}} \cosh\left(\frac{t}{2\mu}\right), \quad u_2 \circ \eta(t, x) = 2\mu \sqrt{\frac{\mu}{r} - 1} \sinh\left(\frac{t}{2\mu}\right), \\ u_3 \circ \xi(t, x) &= u_3 \circ \eta(t, x) = \int_0^r \sqrt{\frac{(h + \mu)(h^2 + \mu^2)}{h^3}} dh \end{aligned}$$

and as before,

$$\begin{aligned} u_4 \circ \xi(t, x) &= u_4 \circ \eta(t, x) = x_1, \quad u_5 \circ \xi(t, x) = u_5 \circ \eta(t, x) = x_2, \\ u_6 \circ \xi(t, x) &= u_6 \circ \eta(t, x) = x_3. \end{aligned}$$

In this case, $(\mathbb{K}_B, \mathbb{K}_N)$ will be called the Fronsdal structure.

Proof. By the condition that $\xi^*(\zeta_U|_{\mathbb{K}_N}) = \zeta_H$, we obtain that

$$\begin{aligned} -(u_1 \circ \xi)_r^2 + (u_2 \circ \xi)_r^2 + (u_3 \circ \xi)_r^2 &= \frac{1}{1 - \mu/r}, \\ -(u_1 \circ \xi)_t^2 + (u_2 \circ \xi)_t^2 + (u_3 \circ \xi)_t^2 &= -\left(1 - \frac{\mu}{r}\right), \\ (u_1 \circ \xi)_t(u_1 \circ \xi)_r + (u_2 \circ \xi)_t(u_2 \circ \xi)_r - (u_3 \circ \xi)_t(u_3 \circ \xi)_r &= 0. \end{aligned}$$

We set $u_4 \circ \xi(t, x) = x_1$, $u_5 \circ \xi(t, x) = x_2$, $u_6 \circ \xi(t, x) = x_3$. Let F, W be mappings from \mathbb{R}^+ into \mathbb{R} and let G, Z be mappings from \mathbb{R} into \mathbb{R} , such that $u_1 \circ \xi(t, x) = G(t)F(r)$, $u_2 \circ \xi(t, x) = Z(t)F(r)$ and $u_3 \circ \xi(t, x) = W(r)$. From the above equations,

$$\begin{aligned} [G(t)^2 - Z(t)^2]' &= 0, \\ [F(r)]^2 [G'(t)^2 - Z'(t)^2] &= 1 - \frac{\mu}{r}, \\ [F'(r)] [Z(t)^2 - G(t)^2] &= \frac{1}{1 - \mu/r} \end{aligned}$$

and the result follows simply by taking $G(t) = \sinh\left(\frac{t}{2\mu}\right)$ and $Z(t) = \cosh\left(\frac{t}{2\mu}\right)$. The proof is same for the black hole. ■

Using the last coordinate equations for ξ and η , one can prove very easily the following Lemma:

Lemma 106 *Assume the hypothesis of Lemma 105. Given a positive real number μ , let $F(\mu)$ be the hypersurface of the Fronsdal manifold U such that $p \in F(\mu)$ if and only if there is some real $r > 0$ for which*

$$\begin{aligned} u_2(p)^2 - u_1(p)^2 &= 4\mu^2 \left(1 - \frac{\mu}{r}\right), \\ u_3(p) &= \int_0^r \sqrt{\frac{(h+\mu)(h^2+\mu^2)}{h^3}} dh, \\ u_4(p)^2 + u_5(p)^2 + u_6(p)^2 &= r^2. \end{aligned}$$

So the Fronsdal structure $(\mathbb{k}_N, \mathbb{k}_B)$ is such that $\mathbb{k}_N \cup \mathbb{k}_B \subset F(\mu)$.

Because of the latter Lemma, $F(\mu)$ will be called the Fronsdal hypersurface with mass μ , while \mathbb{k}_B and \mathbb{k}_N can be identified as the black hole and normal region belonging to the Fronsdal hypersurface.

Exercise 107 *In the notation of Lemma 106, prove that the mapping λ from $F(\mu)$ onto $F(\mu)$ such that $u_1 \circ \lambda(p) = -u_1(p)$, $u_2 \circ \lambda(p) = -u_2(p)$ and $u_i \circ \lambda(p) = u_i(p)$ for all $i \in [3, 6] \subset \mathbb{N}$ is an isometry.*

Thus, by the last Exercise, there are *two* copies of the Hilbert-Droste manifold in the Fronsdal hypersurface, as we would expect from the existence of two “universes” belonging to the Kruskal-Szekeres spacetime.

We are finally motivated to define the maximal extension of the Hilbert-Droste solution from the Fronsdal-Kasner approach:

Definition 108 (Fronsdal spacetime) *The Fronsdal spacetime with mass μ is simply the Fronsdal hypersurface $F(\mu)$ with the metric induced from the vector manifold U (cf. Definition 104).*

Following the reasoning presented in the last section, we could study the geodesics of the Fronsdal spacetime in order to prove that it is indeed maximal. However, it will be more instructive if we explore the relation between the Kruskal-Szekeres spacetime and the former.

In fact, because of Proposition 91 and Lemma 105, we only need to show that the Horizon that connects the black hole and the normal region in the Fronsdal hypersurface is equivalent to that of the Kruskal-Szekeres spacetime.

Definition 109 (Horizon) *Let U be the Fronsdal manifold with natural coordinates (u_1, u_2, \dots, u_6) , $F = F(\mu)$ the Fronsdal spacetime with mass μ and r the mapping from F into \mathbb{R} such that*

$$r(p) = \sqrt{u_4(p)^2 + u_5(p)^2 + u_6(p)^2}.$$

The Horizon \mathcal{H} of F is the submanifold for which $p \in \mathcal{H}$ if and only if $r(p) = \mu$.

Lemma 110 *Let \mathcal{H}_1 and \mathcal{H}_2 be the Horizons of the Kruskal-Szekeres (cf. Definition 86) and Fronsdal spacetimes with mass μ . Then \mathcal{H}_1 and \mathcal{H}_2 are isometric.*

Proof. Let

$$k = \int_0^\mu \sqrt{\frac{(h+\mu)(h^2+\mu^2)}{h^3}} dh.$$

In the notation of Lemma 106, $p \in \mathcal{H}_2$ if and only if $|u_2(p) - u_1(p)| = 0$, $u_3(p) = k$ and $u_4(p)^2 + u_5(p)^2 + u_6(p)^2 = \mu^2$. Let $v_1 = u_1|_{\mathcal{H}}$, ..., $v_6 = u_6|_{\mathcal{H}}$. So

$$dv_1 = dv_2, dv_3 = 0,$$

while

$$dv_4 \otimes dv_4 + dv_5 \otimes dv_5 + dv_6 \otimes dv_6 = \mu^2 \zeta_{S^2},$$

where ζ_{S^2} the Euclidean metric of S^2 . Thus the metric of \mathcal{H} induced from the Fronsdal manifold (cf. Definition 104) becomes

$$-dv_1 \otimes dv_1 + dv_2 \otimes dv_2 + dv_3 \otimes dv_3 + \mu^2 \zeta_{S^2} = \mu^2 \zeta_{S^2}$$

and the result follows from Lemma 87. ■

Hence, it follows from Lemmas 105 and 110 that

Corollary 111 *The Fronsdal spacetime is the embedding in the Fronsdal manifold of the Kruskal-Szekeres spacetime.*

5 Final Considerations

We have explained in details in Section 3 that the solutions of Schwarzschild and Hilbert-Droste are indeed *different* solutions of Einstein equation because they possess a very different topology. And it is important to stress that such a difference is not of a secondary importance since it is possible, in principle, to ascertain which one corresponds to the physical reality by verifying the existence of a black hole in the gravitational field that those solutions describes (that is, in the presence of an isolated point of mass).

Indeed, we have seen that the Schwarzschild solution has as a spacetime manifold the \mathbb{R}^4 with the worldline of the particle generating the field removed, that is, $\mathbb{R} \times \mathbb{R}^+ \times S^2$. Therefore, one cannot have a black hole in such spacetime and, even if a process of maximal extension is formally possible, it would destroy the original topology of the spacetime manifold, which was fixed *a priori* by Schwarzschild (compare this with the example discussed in the last paragraph of this Section). On the other hand, the manifold of the Hilbert-Droste solution have a disconnected topology that is given by $\mathbb{R} \times (\mathbb{R}^+ - \{\mu\}) \times S^2$, which clearly cannot be a satisfactory spacetime model since it is impossible to define a global time orientation in the whole manifold.

Therefore, a maximal extension of the Hilbert-Droste solution is required, as was described in Section 4. There, two approaches were studied. One was the classic Kruskal-Szekeres spacetime, a maximal manifold that contains two regions, one isometric to the black hole and the other to the normal (or exterior) region of the Hilbert-Droste manifold, together with an exotic submanifold

homeomorphic to $[0, \infty[\times S^2$ connecting the black hole to the exterior, generally known as wormhole. The other approach which we covered was the Fronsdal imbedding of the Hilbert-Droste “spacetime” in a 6-dimensional vectorial manifold by improving a procedure minted by Kasner almost four decades before.

On what follows, we shall finish our endeavour commenting some works in the literature.

We begin with an author who as able to appreciate most of what was told above, the differential geometer N. Stavroulakis. He wrote a series of notes entitled *Vérité scientifique et trous noirs* [5], published in the *Annales de la Fondation Louis de Broglie* in 1999. There, the geometer presented a critical analysis of many practices usually employed by relativists when studying solutions of Einstein equation with black holes.

For instance, recognizing that solutions with different manifolds actually describes different physical situations, Stavroulakis wrote in his notes that

“Puisque la variété n’est pas fixée d’avance, la présentation d’une solution dans divers systèmes de coordonnées locales dissimule souvent l’utilisation de variétés différentes. Mais alors il s’agit d’un problème sans objet, car l’introduction de variétés distinctes donne lieu nécessairement à des problèmes distincts”.

Stavroulakis was particularly discontented with the use of manifolds with boundary and the use of what he called “implicit transformations” when solving Einstein equation.

Concerning the first, the geometer criticizes, for instance, the continuity condition that Schwarzschild adopted to determine one of the constants which appears in his solution (cf. Remark 67). Indeed, such a condition requires the introduction of the manifold with boundary $\mathbb{R} \times [0, \infty[\times S^2$, so that the metric tensor is required to be continuous in the whole new spacetime manifold except at the boundary $\{0\} \times S^2$.

However, since $\mathbb{R} \times [0, \infty[\times S^2$ is not homeomorphic to \mathbb{R}^4 nor to any submanifold of the latter, its boundary $\{0\} \times S^2$ is absent of physical meaning, if one actually *believes* that the gravitational field of an isolated mass point must be described by some submanifold of \mathbb{R}^4 . This is not the case of course of the Kruskal-Szekeres spacetime.

Another criticism of the use of manifolds with boundary stressed by Stavroulakis is that, in the case of the Schwarzschild manifold, a Riemannian “metric” defined on $\mathbb{R} \times [0, \infty[\times S^2$ which is continuous (differentiable, respectively) and positive in $\mathbb{R} \times]0, \infty[\times S^2$, is however null at the boundary $\{0\} \times S^2$ – something which we may call a *pseudo-metric* – normally lead to a metric which is not continuous (differentiable, respectively) at the origin of \mathbb{R}^4 . A simple example can be given in $[0, \infty[\times S^2$ with a *pseudo-metric* defined by

$$2dr \otimes dr + r^2 \zeta_{S^2}$$

where r is the identity of $[0, \infty[$ and ζ_{S^2} the Euclidean metric of S^2 . Introducing

natural coordinates $(x^i)_{i \in [1,3]}$ in \mathbb{R}^3 , the metric induced in $\mathbb{R}^3 - \{0\}$ is

$$\sum_{i \in [1,3]} dx^i \otimes dx^i + \frac{1}{\|x\|^2} \left(\sum_{i \in [1,3]} x^i dx^i \right) \otimes \left(\sum_{i \in [1,3]} x^i dx^i \right)$$

where $\|x\|^2 = \sum (x^i)^2$, which of course is undefined for $0 \in \mathbb{R}^3$. The same can be easily generalized for the manifold of interest, $\mathbb{R} \times [0, \infty[\times S^2$. For example, the reader may verify that, given mappings f, g from $\mathbb{R} \times [0, \infty[\times S^2$ into \mathbb{R} , both differentiable in $\mathbb{R} \times]0, \infty[\times S^2$, if we define in $\mathbb{R} \times [0, \infty[\times S^2$ the Bondi's pseudo-metric,

$$- \exp(2f) dt \otimes dt - \exp(f + g) (dt \otimes dr + dr \otimes dt) + r^2 \zeta_{S^2}$$

where (t, r) is the natural coordinate system of $\mathbb{R} \times [0, \infty[$ and ζ_{S^2} as before, then, after expressing the latter metric in the natural coordinates of \mathbb{R}^4 , the same cannot be defined for $0 \in \mathbb{R}^4$.

So Stavroulakis argues

“Dans de telles situations la différentiabilité de la forme considérée sur $[0, \infty[\times S^2$ est illusoire, car elle dissimule les singularités de la forme d'origine sur \mathbb{R}^3 . La géométrie différentielle classique ne prend pas en considération les situations de ce genre qui nécessitent une étude à part afin d'élucider la nature des singularités. En ce qui concerne la relativité générale, on ne saurait introduire des métriques comportant des singularités génériques”.

However, at this point, one may take a position different from that of Stavroulakis and argue that the Schwarzschild solution is a reasonable physical model, even with the Schwarzschild's continuity condition making reference to the boundary $\{0\} \times S^2$. That is because the “singularity” that such condition introduces in the solution (in $\mathbb{R} \times \mathbb{R}^3$) is along the worldline of the particle generating the field, what may be seen as physically acceptable.

The situation here is similar to classical mechanics, when one removes a finite set of points in \mathbb{R}^3 in order to deal with problems involving particles interacting through a Newtonian potential. (Indeed, as it is well known, the only situation that such a description encounters difficulty in classical mechanics is in the presence of collisions). But we shall not enter on this discussion here and the interested reader must consult [5].

Remark 112 *As Stavroulakis himself believed that the solution which describes the gravitational field of a spherically symmetric body must have $\mathbb{R} \times \mathbb{R}^3$ as spacetime manifold, he proposed his own solution in [42], where he argues that the existence of point of mass is a hypothesis incompatible with General Relativity on the grounds that such objects introduces singularities in the solution.*

On the issue of the “implicit transformations”, Stavroulakis said that

“Une transformation implicite est censée être définie par un système d’équations (équations ordinaires pour la définition de fonction implicites, équations différentielles, équations aux dérivées partielles) contenant les composantes inconnues du tenseur métrique (...). Or la solution effective d’un tel système ne pourrait être envisagée que si les composantes en question étaient connues. Par conséquent les transformations implicites sont des transformations hypothétiques dont l’existence même sur U (ou éventuellement sur un ouvert contenu dans U) n’est pas assurée”.

The “implicit transformations”, defined by Stavroulakis in the above excerpt, were employed in the construction of the Hilbert-Droste solution, as the reader can recall from Subsection 3.3. Indeed, the special coordinates (t, h) of the Schwarzschild plane Π_M (check Definitions 55 and 44) were chosen to be such that the metric component α in the warped product $\Pi_M \times_\alpha S^2$ obeys the condition $\alpha \circ h = \text{id}_{\mathbb{R}}$. That is, our coordinate system was chosen in such a way that the metric could be writing as

$$- (f \circ h) dt \otimes dt + (g \circ h) dh \otimes dh + h^2 \zeta_{S^2} \quad (12)$$

because $h \mapsto \alpha(h) = h$. So, according to Stavroulakis, this implicit transformation is the origin of the famous singularity at $h = \mu$. As he wrote in [43] (with our notation and enumeration)

“As is expected, the solution of the Einstein equations related to 12 is static:

$$- \left(1 - \frac{\mu}{h}\right) dt \otimes dt + \frac{1}{1 - \mu/h} dh \otimes dh + h^2 \zeta_{S^2}$$

In fact it is the Droste solution, or, more precisely, the Droste-Hilbert solution, wrongly called Schwarzschild’s solution in the literature. We have already seen that the implicit diffeomorphism considered (...) is in general actually inexistent. Now the discontinuity of the Droste solution at $h = \mu$ proves that the implicit diffeomorphism in question is also inconsistent with the differentiable solutions of the Einstein equations”.

And, as repeated many times in our work, the “singularity $h = \mu$ ” causes the manifold of the Hilbert-Droste solution to be disconnected, so that we can conclude that the use of an implicit transformation is the origin of the necessity of the maximal extension of the Hilbert-Droste solution, the Kruskal-Szekeres spacetime.

As a matter of fact, the maximal extension was also a target of Stavroulakis criticism, as we can see from [5],

“L’introduction des variétés à bord à entraîné l’idée bizarre d’extension maximale. Celle-ci est vide de sens par rapport à la variété $\mathbb{R} \times \mathbb{R}^3$.”

En ce qui concerne l’extension de $\mathbb{R} \times [0, \infty[\times S^2$ au sens de Kruskal, elle nécessite des identifications au moyen d’applications discontinues qui ne sont pas mathématiquement autorisées”.

or, from the same source,

“La méthode de Kruskal elle-même comporte des incohérences qui transgressent les principes élémentaires des raisonnements mathématiques”.

We, of course, cannot agree with the idea that the maximal extension of the Hilbert-Droste solution is mathematically inconsistent, since we have dedicated Section 4 to two distinct approaches to the extension of the Hilbert-Droste manifold, that of Kruskal and Szekeres and the imbedding of Kasner and Fronsdal. So now we must understand on what grounds Stavroulakis based his latter remarks.

In fact, the geometer was referring to the idea of an “apparent singularity”, disseminated in the Relativity community and whose germ can be found in the original papers of Synge, Kruskal and Szekeres, as, e.g., one can see from Synge’s letter to the editor [37], partly quoted above, (in our notation)

“The existence of the singularity at $h = \mu$ is strange and demands investigation. Investigation shows that there is no singularity at $h = \mu$; the apparent singularity in the Schwarzschild line element is due to the coordinates employed and may be removed by a transformation, so that there remains no singularity except at $h = 0$. This was pointed out by Lemaitre in 1933”.

A singularity is classified as “apparent” if it can be attributed to a bad choice of coordinate system, and can be introduced or removed from a coordinate expression of the metric through a coordinate change. However, as Stavroulakis argued, this kind of transformation cannot be a diffeomorphism, being therefore not a permissible coordinate transformation.

For instance, from a formal point of view, the “singularity $h = \mu$ ” of the Hilbert-Droste solution cannot be removed by means of a coordinate transformation like Equations 3 and 4 used in the proof of Lemma 79 or the one used originally by Kruskal, given by Equations 10 and 11, since that both transformations are degenerated exactly at, in the notation of this section, $h = \mu$.

What we can conclude from Stavroulakis remarks, however, is that, from a formal perspective, the maximal extension of a solution of Einstein equation is not executed by means of a coordinate change, but by *postulating* a manifold, like the Kruskal-Szekeres manifold, and showing that there exists submanifolds belonging to the former which are isometric to the submanifolds of the solution in question, e.g., the black hole and the normal region of the Hilbert-Droste “spacetime”.

On the other hand, not everyone understood so well the issues presented in our work. For instance, referring to some modern concepts in General Relativity, like “event horizon”, “essential singularity” and, we empathize, “*maximal extension*”, Ll. Bel wrote in [3] that

“All this is nice geometry in the making but the point is that none of this is as yet necessary to understand that Schwarzschild’s original work is a better piece of physics than the extravaganzas to which one is led with some of the extensions of Schwarzschild’s solution”.

If Ll. Bel was referring to the original Schwarzschild solution, as he claim to be, then he would be completely right since the spacetime manifold of the latter solution is entirely satisfactory on its own, being connected and dispensing any process of maximal extension. Unfortunately, however, as Bel confuses the original “Schwarzschild” solution with that of Hilbert and Droste, we believe that he must be wrong because, as stressed many times in our work, the Hilbert-Droste manifold cannot define a proper spacetime – in the sense of Definition 20 (see also Remark 21 for a motivation of the latter Definition) – and therefore, a maximal extension is not any “extravaganza”, but a necessary procedure if one is willing to accept the Hilbert-Droste “spacetime” as a description of Nature.

Indeed, comparing the coordinate expressions for the metric in the Schwarzschild solution to the one in the Hilbert-Droste solution, Bel wrote in that same article that

“This new form [*the Hilbert-Droste metric*] is simpler to obtain than (1) [*the Schwarzschild metric*] and also simpler to write down and is the form which is used overwhelmingly in textbooks. Notice that it can be derived directly from Schwarzschild’s form following two different, but equivalent, paths:

- (i) To use the definition (3)¹ of the auxiliary function R as a coordinate transformation and get rid of the spurious parameter, or
- (ii) choose for simplicity $\rho = 0$ ²”.

Then he explains why Schwarzschild could not follow his “path (i)”,

“Schwarzschild could not follow the first path because he thought he was dealing with a theory which did not allow arbitrary coordinate transformations (but in fact he had already done it when he abandoned his initial coordinates for those used in (1))”.

And Bel is not alone in his opinion. As P. Fromholz *et al.* wrote in [4],

“Schwarzschild noticed that by defining a new variable

$$r_s \equiv (3x + b)^{1/3} = (\rho^3 + b)^{1/3}$$

he could put the metric (6) [*referring to Schwarzschild metric*] into a simpler form, which is precisely Eq. (4) [*the Hilbert-Droste metric*]”.

¹In our notation, $R \equiv \alpha(h) = (3h + \mu^3)^{1/3}$. Recall Proposition 68.

²In the notation of Proposition 65, $k = 0$.

But the authors of the latter article went a little further showing a complete ignorance about the topology which Schwarzschild himself fixed in his spacetime manifold, writing that

“But Schwarzschild went on to address the integration constant b . He demanded that the metric be regular everywhere except at the location of the mass-point, which he assigned to be at $\rho = 0$, where the metric should be singular. This fixed $b = (2M)^3$. This choice resulted in considerable confusion about the nature of the “Schwarzschild singularity”, which was not cleared up fully until the 1960s. Because we now are attuned to the complete arbitrariness of coordinates, we understand that $\rho = 0$, or $r_s = 2M$ is not the origin, but is the location of the event horizon, while $\rho = -2M$, or $r_s = 0$ is the location of the true physical singularity inside the black hole”.

Observe that Fromholz *et al.* talks about an “event horizon” and a “true physical singularity inside the black hole” in a solution whose manifold is given by $\mathbb{R} \times \mathbb{R}^+ \times S^2$, which is already a nonsense. But this is not the worse part yet. Now, when these authors wrote “ $\rho = -2M$ ”, they completely disrespected the topology (and even the domain of definition of the chart) chosen by Schwarzschild in his original paper, since that x (in Fromholz *et al.* notation and x_1 according to Schwarzschild paper), representing the radial coordinate of $\mathbb{R} \times \mathbb{R}^+ \times S^2$ – in fact, $x = r^3/3$ – must be a positive real number, so that $\rho = \sqrt[3]{3x} > 0$.

Remark 113 *It is worth mentioning that Schwarzschild could not choose $k = 0$ (our notation; see footnote of latter page), as suggested by Bel, because it would give a solution incompatible with the manifold fixed by Schwarzschild. That is because, with the condition $k = 0$, the coordinate expression for the metric would have a singularity in $R = \mu$ (our notation), something that can be interpreted in two ways. First, if such a singularity is seen as a property of the metric tensor, it would not be satisfactory because the set of points for which $R = \mu$ is contained in the Schwarzschild manifold. Second, if the singularity is interpreted as “apparent” because of a bad choice of a coordinate system (the usual perspective today), then the spacetime manifold of the solution could not be $\mathbb{R} \times \mathbb{R}^+ \times S^2$ because, as it was proved in Section 4, the maximal extension of such a solution (the Kruskal-Szekeres spacetime) have an exotic topology totally different from $\mathbb{R} \times \mathbb{R}^+ \times S^2$.*

J. Senovilla left a similar opinion in his rectification note [2] concerning the equivalence of Schwarzschild solution and that of Hilbert-Droste:

“I would like to remark here that Karl Schwarzschild did write the form (1) [referring to the Hilbert-Droste coordinate expression] of the metric: see formula (14) in the GRG Golden Oldie translation [16](b). The myth that he did not do it must be dispelled. To argue

that the R in that formula was in fact a function of the radial coordinate that he used – due to the famous story of the unit-determinant gauge choice favoured by Einstein at early stages – is completely irrelevant today, given the general covariance of the theory and, especially, the fact that Schwarzschild wrote “ dR^2 ” and expressed the whole line-element in terms of R exclusively.”

However, the fact that General Relativity is “generally covariant” or, in more precise terms, diffeomorphic invariant, does not mean that one can arbitrarily change the topology of the spacetime manifold, but only the coordinate system. That is, even if there exists a coordinate transformation (which was described here in Remark 66) that transforms the coordinate expression for the Schwarzschild metric to one with the same form as the Hilbert-Droste expression, one cannot jump to the conclusion that these solutions are indeed the same.

And, differently from some of the latter authors, if we pay attention to the domains of the coordinate transformation of Remark 66, it must be clear that the diffeomorphism $h \mapsto R = \alpha(h) = (3h + \mu^3)^{1/3}$, from $]0, \infty[$ onto $] \mu, \infty[$, transforms the coordinate expression of the Schwarzschild metric (notation as in Section 3)

$$- \left[1 - \frac{\mu}{\alpha(h)} \right] dt \otimes dt + \frac{1}{[\alpha(h)]^4} \frac{1}{1 - \mu/\alpha(h)} dh \otimes dh + [\alpha(h)]^2 \zeta_{S^2}$$

to one reassembling the Hilbert-Droste metric,

$$- \left(1 - \frac{\mu}{R} \right) dt \otimes dt + \frac{1}{1 - \mu/R} dR \otimes dR + R^2 \zeta_{S^2}$$

which holds, however, only for $R > \mu$, since that $\alpha(]0, \infty[) =] \mu, \infty[$. That is, the “interior” $R < \mu$ is meaningless in the Schwarzschild solution, as it describes a manifold which is completely disconnected from the former.

Remark 114 *The reason for which the diffeomorphism $h \mapsto R = \alpha(h)$ was defined on $]0, \infty[$ is that h belongs to the natural coordinate system of what we called above the Schwarzschild plane $P = \mathbb{R} \times \mathbb{R}^+$. Indeed, h can be interpreted as the radial coordinate of the Schwarzschild spacetime manifold $\mathbb{R} \times \mathbb{R}^+ \times S^2$, which is possible since Schwarzschild fixed his manifold **a priori**.*

We finish by discussing a trivial example that illustrates very well what was told above and can be found in the same paper where Szekeres presented his maximal extension of the Hilbert-Droste solution (cf. ref. [25]), almost five decades ago. Let $\mathbb{R}^+ \times S^2$ be given with the structure of an Euclidean manifold, that is, with the metric given by

$$g = dh \otimes dh + h^2 \zeta_{S^2}$$

where h is the identity (or the natural coordinate) of \mathbb{R}^+ and ζ_{S^2} is the Euclidean metric of S^2 . Let $\mu > 0$ be a real number. So, with the diffeomorphism $h \mapsto$

$R = h + \mu$ from \mathbb{R}^+ onto $] \mu, \infty[$, the coordinate expression for g can be written as

$$g = dR \otimes dR + (R - \mu)^2 \zeta_{S^2} \quad (13)$$

which holds, *as in the Schwarzschild case*, only for $R > \mu$. That is, the Euclidean space which we started with is now identified with $R > \mu$, in such a way that speak about the “region” $R < \mu$ here – without changing our original manifold – is simply a *nonsense*, because it is not only outside the domain of definition of our diffeomorphism, as it describes a submanifold which is disconnected from $\mathbb{R}^+ \times S^2$, having nothing to do with our ordinary Euclidean topology. As Szekeres remarked in ref. [25] (with our notation),

“Here we have an apparent singularity on the sphere $R = \mu$, due to a spreading out of the origin over a sphere of radius μ . Since the exterior region $R > \mu$ represents the whole of Euclidean space (except the origin), the interior $R < \mu$ is entirely disconnected from it and represents a distinct manifold”.

If one insists, however, we can still use the expression for the metric given by Eq. (13) for all $R \in]0, \infty[- \{\mu\}$, but only *if we pay the price of changing our original manifold*. That is, we may define a new manifold $M = (\mathbb{R}^+ - \{\mu\}) \times S^2$, which is not equivalent to the Euclidean $\mathbb{R}^+ \times S^2$, and give to it a Riemannian structure whose metric is defined by

$$g' = dR \otimes dR + (R - \mu)^2 \zeta_{S^2}$$

for all positive real number $R \neq \mu$. The fact that the Riemannian structures $(\mathbb{R}^+ \times S^2, g)$ and (M, g') are different from each other (even if the former contains a submanifold isometric to the latter) are very far from being controversial – since there is no black hole or any other polemical issue in the present game – but is, on the other hand, of the same nature as the difference between the solutions of Schwarzschild and Hilbert-Droste, which are polemical subjects in the current literature.

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A Topological Extension of Manifolds

In this Appendix, we analyze the extension of manifolds from a careful topological point of view. Specifically, we give a rigorous procedure (summarized in the following list of Definitions and Lemmas) that justify the process of gluing topological spaces, manifolds and pseudo-Riemannian structures.

We hope that the following developments might be useful for relativists working in the construction of spacetimes containing black holes, wormholes, bridges or any object with exotic topology.

Our approach is based in ref. [13].

Definition 115 *A gluing structure is an ordered list (M, N, U, V, ξ) , where M and N are topological spaces, $U \subset M$ and $V \subset N$ are subspaces and ξ is a homeomorphism between U and V .*

Recall that, given a family of sets $(A_n)_{n \in F}$, the disjoint union of this family is defined to be

$$\tilde{\cup}_{n \in F} A_n = \cup_{i \in F} \{(x, i) : x \in A_i\}$$

Definition 116 *Let $G = (M, N, U, V, \xi)$ be a gluing structure. Define \approx to be the equivalence relation on the disjoint union $M \tilde{\cup} N$ such that $p \approx q$ if and only if $p = q$, $p = \xi(q)$ or $q = \xi(p)$. So the quotient space $M \tilde{\cup} N / \approx$ will be called the glued space Q_G of G and \approx the equivalence of the gluing structure G .*

In what follows, given a gluing structure $G = (M, N, U, V, \xi)$ and its respective glued space Q_G , the natural injections i and j from G into Q_G are the mappings from M and N , respectively, into Q_G such that

$$\begin{aligned} i(p) &= p \text{ if } p \in M - U \text{ and } i(p) = \{p, \xi(p)\} \text{ if } p \in U, \\ j(q) &= q \text{ if } q \in N - V \text{ and } j(q) = \{q, \xi(q)\} \text{ if } q \in V. \end{aligned}$$

A subset $S \subset Q_G$ will be considered open if and only if $i^{-1}(S)$ and $j^{-1}(S)$ are open in M and in N , respectively.

Lemma 117 *Let G be a gluing structure, Q_G its glued space and i and j the natural injections from G into Q_G . Then i and j are homeomorphisms between M and $i(M)$ and between N and $j(N)$, respectively.*

Proof. By the last remark, i and j are continuous. Let $X \subset M$ be an open subset. So $i(X)$ is open in Q_G if and only if $i^{-1}(i(X))$ and $j^{-1}(i(X))$ are open in M and in N respectively. The first is open since that $i^{-1}(i(X)) = X$. But for the second:

$$j^{-1}(i(X)) = j^{-1}(i(X) \cap j(N)) = j^{-1}(i(X \cap U)) = \xi(X \cap U)$$

Hence $i(X)$ is open. The result follows for i since that it is injective, and the proof is the same for j . ■

Remark 118 *Because of the last Lemma, one may ignore the natural injections and think about $i(M)$ and $j(N)$ as being actually equal to M and N , respectively. Then $M \cap N$, U and V are all identified.*

Lemma 119 *Assuming the hypothesis of Lemma 117, let P be a topological space and let ϕ_M and ϕ_N be continuous mappings from M and N , respectively, into P . Suppose that $\phi_M|_U = \phi_N \circ \xi$. Hence, there is a unique continuous mapping ϕ from Q_G into P such that $\phi \circ i = \phi_M$ and $\phi \circ j = \phi_N$.*

Proof. Define ϕ to be such that $\phi(p) = \phi_M(i^{-1}(p))$ if $p \in i(M)$ and $\phi(q) = \phi_N(j^{-1}(q))$ if $q \in j(N)$. This is well-defined since that when $p \in i(M) \cap j(N)$, $p = \{x, \xi(x)\}$ for $x = i^{-1}(x)$. Hence

$$\phi(p) = \phi_M(x) = \phi_N(\xi(x)) = \phi(p)$$

Finally, ϕ is continuous by Lemma 117. ■

Lemma 120 Let $G = (M, N, U, V, \xi)$ and $G' = (M', N', U', V', \xi')$ be gluing structures, Q_G and $Q'_G = Q_{G'}$ their respective glued spaces and i, j and i', j' their respective natural projections. Let ϕ_M and ϕ_N be continuous mappings from M and N , respectively, into M' and N' , respectively. Assume that $\xi' \circ \phi_M|U = \phi_N \circ \xi$. Thus, there is a unique continuous mapping ϕ from Q_G into Q'_G such that $\phi \circ i = i' \circ \phi_M$ and $\phi \circ j = j' \circ \phi_N$.

Proof. Define ϕ to be such that $\phi(p) = i'(\phi_M(i^{-1}(p)))$ if $p \in i(M)$ and $\phi(q) = j'(\phi_N(j^{-1}(q)))$ if $q \in j(N)$. This is well-defined since that, if $p \in i(M) \cap j(N)$, $p = \{x, \xi(x)\}$ for $x = i^{-1}(p)$. So

$$\phi(p) = i'(\phi_M(x)) = (j' \circ \xi')(\phi_M(x)) = j'(\phi_N(\xi(x))) = \phi(p)$$

Finally, ϕ is continuous by Lemma 117. ■

The last two Lemmas are normally called the *Mapping Lemmas*.

Exercise 121 (a) Let $G = (\mathbb{R}, \mathbb{R}, \mathbb{R}^+, \mathbb{R}^+, \text{id}_{\mathbb{R}})$ be a gluing structure. (For any set A , id_A means the identity in A). Is Q_G , the glued space, Hausdorff? (b) Let H^+ be the north hemisphere of S^2 without the equator, and let $N \in S^2$ be its north pole. Let $G = (S^2, S^2, H^+ - \{N\}, H^+ - \{N\}, \text{id}_{S^2})$ be a gluing structure. Is Q_G Hausdorff?

The above exercise is then the motivation for the following definition:

Definition 122 A gluing structure (M, N, U, V, ξ) will be called Hausdorff if M and N are Hausdorff and if there is no convergent sequence $(p_n)_{n \in \mathbb{N}}$ of points in M such that $\lim p_n \in M - U$ and $\lim \xi(p_n) \in N - V$.

Lemma 123 Let (M, N, U, V, ξ) be a Hausdorff gluing structure. So the glued space Q_G is Hausdorff.

Proof. Let $x, y \in Q_G$ be distinct points. The result is obvious if both x and y belongs to $i(M)$ (or to $j(N)$). So, suppose that $x \in i(M) - j(V)$ and $y \in j(N) - i(U)$. Let $(N_n)_{n \in \mathbb{N}}$ and $(N'_n)_{n \in \mathbb{N}}$ be a basis for the neighborhoods of x and y , respectively. Assume that $N_n \cap N'_n$ is not empty for all n . So by the axiom of choice, there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in N_n \cap N'_n$ for all n . Let i and j be the natural injections of G into Q_G . So $(i^{-1}(x_n))_{n \in \mathbb{N}}$ and $(j^{-1}(x_n))_{n \in \mathbb{N}}$ do not respect Definition 122. Hence, by contradiction, there is some $n \in \mathbb{N}$ such that $N_n \cap N'_n = \emptyset$, and the proof is over. ■

Exercise 124 Let $U = V = \{(x, y) \in \mathbb{R}^2 : x, y < 0\}$ and $G = (\mathbb{R}^2, \mathbb{R}^2, U, V, \xi)$. Is the glued space Q_G Hausdorff if (a) $\xi = \text{id}_{\mathbb{R}^2}$ and (b) $\xi(x, y) = (x, y/x)$?

We can now extrapolate our results for manifolds:

Definition 125 A gluing structure $G = (M, N, U, V, \xi)$ will be called a “manifold gluing” when G is Hausdorff, M and N are manifolds with the same dimension, U and V are submanifolds and ξ is a diffeomorphism.

Remark 126 It must be clear from the above definition that, in the “manifold gluing” case, the Mapping Lemmas holds for smooth mappings rather than just for continuous ones.

Remember that a chart in a manifold M is an ordered pair (X, ψ) such that $X \subset M$ is an open subset and ψ is a homeomorphism between X and $\mathbb{R}^{\dim M}$. Recall also that an atlas in M is a set A of charts such that $M \subset \bigcup_{(U, \psi) \in A} U$ (we say that A covers M) and, given two charts $(X, \psi), (Y, \omega) \in A$ such that $X \cap Y \neq \emptyset$, both $\psi \circ \omega^{-1}$ and $\omega \circ \psi^{-1}$ are smooth (we say that A overlaps smoothly).

Lemma 127 Let $G = (M, N, U, V, \xi)$ be a “manifold gluing”. The glued space Q_G is itself a manifold.

Proof. By hypothesis, Q_G is Hausdorff. Now, let A_M and A_N be atlases for M and N respectively, and define

$$A = \{(i(X), \psi \circ i^{-1}), (j(Y), \omega \circ j^{-1}) : (X, \psi) \in A_M, (Y, \omega) \in A_N\}.$$

Of course that A covers Q_G . To prove that they overlap smoothly, let $(X, \psi) \in A_M$ and $(Y, \omega) \in A_N$ be charts such that $i(X) \cap j(Y) \neq \emptyset$. So

$$(\psi \circ i^{-1}) \circ (\omega \circ j^{-1})^{-1} = \psi \circ (i^{-1} \circ j) \circ \omega^{-1}$$

is smooth and the proof is over. ■

Finally, we finish with the pseudo-Riemannian case:

Definition 128 A Hausdorff gluing structure $G = (M, N, U, V, \xi)$ will be called a pseudo-Riemannian gluing if M and N have pseudo-Riemannian structures (see Definition 1) and if ξ is an isometry.

Proposition 129 Let $G = (M, N, U, V, \xi)$ be a pseudo-Riemannian gluing and g_M and g_N the metric tensors of M and N , respectively. So, there is a unique metric tensor g_G such that (Q_G, g_G) is a pseudo-Riemannian manifold.

Proof. Let i and j be the natural projections of G into Q_G and let $V, W \in \sec TQ_G$. So the mappings $x \rightarrow \phi_M^{(V,W)}(x) = g_M(i_*^{-1}(V_x), i_*^{-1}(W_x))$, from M into \mathbb{R} , and $y \rightarrow \phi_N^{(V,W)}(y) = g_N(j_*^{-1}(V_y), j_*^{-1}(W_y))$, from N into \mathbb{R} , are smooth. By the *Mapping Lemmas* and Remark 126, there is a unique smooth mapping $p \rightarrow \phi^{(V,W)}(p)$ from Q_G into \mathbb{R} such that $\phi^{(V,W)} \circ i = \phi_M^{(V,W)}|_U$ and $\phi^{(V,W)} \circ j = \phi_N^{(V,W)}|_V$. Hence, just define $g_G \in \sec T^2Q_G$ to be such that $g_G(V_p, W_p) = \phi^{(V,W)}(p)$ and it is left to the reader to show why g_G is smooth. ■

Thus the title of this Appendix is justified by the fact that Q_G can be called, suggestively, the extension of the manifolds M and N .

B Einstein-Rosen Bridge

In this Appendix, we shall comment briefly on the mathematical realization of the Einstein-Rosen bridge (ER bridge for short). Also, the well-known fact that one cannot “travel” through that bridge, and even from one “universe” of the Kruskal-Szekeres spacetime to another (i.e., from \mathcal{R}_I^+ to \mathcal{R}_I^- or from \mathcal{R}_{II}^+ to \mathcal{R}_{II}^-), will be proven.

A. Einstein and N. Rosen published in 1935 an article entitled “*The Particle Problem in the General Theory of Relativity*” (cf. ref. [26]). There, the authors proposed to eliminate the “ $r = \mu$ ” singularity of the Hilbert-Droste solution by introducing the idea that elementary particles, in particular the electron, are an exotic topological deformations of the spacetime manifold.

The traditional and heuristic construction of the ER bridge, which can be found in any standard text on the subject of wormholes (cf. ref. [27]), proceeds as follows. One starts with the HD manifold $M = \mathbb{R} \times (\mathbb{R}^+ - \{\mu\}) \times S^2$ for some positive real μ and define on it the HD metric

$$-\left(1 - \frac{\mu}{r}\right) dt \otimes dt + \frac{1}{1 - \mu/r} dr \otimes dr + r^2 \zeta_{S^2},$$

where (t, r) are natural coordinates of \mathbb{R}^2 restricted to M and, as usual, ζ_{S^2} is the Euclidean metric of S^2 . Now, the “coordinate transformation” $r \mapsto u^2 = r - \mu$ is introduced, and it is *claimed* that the above metric can be translated to

$$\zeta_{ER} = -\frac{u^2}{u^2 + \mu^2} dt \otimes dt + 4(u^2 + \mu^2) dr \otimes dr + (u^2 + \mu^2) \zeta_{S^2},$$

holding for all $u \in \mathbb{R}$ while $r \in [\mu, \infty[$. Or, in M. Visser words [27] (preserving his notation),

“This coordinate change discards the region containing the curvature singularity $r \in [0, 2M)$, and twice covers the asymptotically flat region, $r \in [2M, \infty)$. The region near $u = 0$ is interpreted as a “bridge” connecting the asymptotically flat region near $u = +\infty$ with the asymptotically flat region near $u = -\infty$ ”.

From a mathematical point of view, this is of course a *non sequitur* since $r \mapsto u^2 = r - \mu$ is not a diffeomorphism $[\mu, \infty[\rightarrow \mathbb{R}$.

We can make the construction of the ER bridge precise as follows. Let³

$$\mathfrak{B} = \mathbb{R} \times S^2 \approx \mathbb{R} \times \{0\} \times S^2,$$

$$\mathcal{N}_1 = \mathbb{R} \times [\mu, \infty[\times S^2, \quad \mathcal{N}_2 = \mathbb{R} \times]-\infty, -\mu] \times S^2.$$

Define on these manifolds the pseudo-Riemannian structures $(\mathcal{N}_1, \zeta_{ER}|_{\mathcal{N}_1})$, $(\mathcal{N}_2, \zeta_{ER}|_{\mathcal{N}_2})$ and $(\mathfrak{B}, \zeta_{ER}|_{\mathfrak{B}})$. As \mathfrak{B} can be identified with the boundaries of \mathcal{N}_1 and \mathcal{N}_2 , letting $\text{id}_{\mathfrak{B}} : \mathfrak{B} \rightarrow \mathfrak{B}$ be the identity mapping, we can state our

Definition 130 *Let $G = (\mathcal{N}_1, \mathcal{N}_2, \mathfrak{B} \subset \mathcal{N}_1, \mathfrak{B} \subset \mathcal{N}_2, \text{id}_{\mathfrak{B}})$ be a pseudo-Riemannian gluing. Then the ER spacetime manifold is the glued space Q_G , \mathfrak{B} is called the ER bridge and \mathcal{N}_1 and \mathcal{N}_2 the exterior regions.*

In order to understand the bridge geometry, we state our

Proposition 131 *The ER bridge \mathfrak{B} is mapped onto S^2 by a homothety with coefficient μ .*

Proof. The restriction $\zeta_{ER}|_{\mathfrak{B}}$ is $4\mu^2 dr \otimes dr + \mu^2 \zeta_{S^2}$ since $u|_{\mathfrak{B}} = 0$. But because $r|_{\mathfrak{B}} = \mu$, $d(r|_{\mathfrak{B}}) = 0$. Thus the former metric becomes $\mu^2 \zeta_{S^2}$. ■

It is necessary some care in order to interpret the latter Proposition. As it was defined above, the bridge \mathfrak{B} has the topology of $\mathbb{R} \times S^2$, where the real line \mathbb{R} represents physically the time. However, since the metric degenerates on \mathfrak{B} in such a way that the metric component accompanying $dt \otimes dt$ vanishes, the natural projection $\mathbb{R} \times S^2 \rightarrow S^2$, $(t, \theta) \mapsto \theta$ becomes a homothety when applied to the metric tensor of \mathfrak{B} , mapping \mathfrak{B} as a pseudo-Riemannian structure onto the 2-dimensional sphere with radius μ .

In this sense, one might think about the solution mass μ as being the radius of the “throat” of the ER bridge.

Now we devote some words to comment on the relation of the Horizon living in the Kruskal-Szekeres spacetime to the ER Bridge. In his original work, Kruskal himself (cf. ref. [24]) understood the Horizon as a kind of bridge or a “wormhole” in the sense of Misner and Wheeler. And indeed, from a mathematical viewpoint, Lemma 87 and Proposition 131 shows that the Horizon and the ER bridge not only shares the same topology as they are both mapped onto S^2 by a homothety, whose coefficient equals the solution mass.

On the other hand, it is clear that the construction of the Horizon can be regarded as more “natural” than the ER bridge, in the sense that while the latter is based on an *ad hoc* gluing of manifolds, the former is a necessary consequence of the maximal extension of the Hilbert-Droste manifold.

³ \approx means: homeomorphic to.

However, we shall admire the creativity and the originality of Einstein and Rosen in anticipating some aspects of the KS spacetime three decades before the publications of Kruskal and Szekeres.

We shall finish this Appendix showing that it is not a good idea to regard the Horizon or even the bridge \mathfrak{B} as a “wormhole” since it is impossible to use these objects to travel from one region to another.

To prove this, it is clearly sufficient to consider only the motion of lightlike geodesics⁴. Let γ be a null geodesic into \mathcal{R}_{II}^+ ending in the Horizon (or the ER bridge). So by Lemma 90, there exists some $\varepsilon \in \{-1, +1\}$ and a reparametrization of γ , say, $\{s \in \mathbb{R} : \varepsilon s > \mu\} \xrightarrow{\gamma} \mathcal{R}_{II}^+$, such that for some HD coordinates,

$$r \circ \gamma(s) = \varepsilon s, \quad t \circ \gamma(s) = s + \varepsilon \mu \log |\mu - \varepsilon s|.$$

Since we are interested in in-going geodesics, that is, geodesics which falls in the Horizon or in the ER bridge, we must choose $\varepsilon = -1$.

Using the diffeomorphism $\mathbb{R} \times]\mu, \infty[\xrightarrow{\xi} \mathcal{R}_{II}^+$ whose existence is ensured by Lemma 79, we can rewrite the above geodesic parametrization in the Kruskal-Szekeres coordinates. That is, letting

$$\begin{aligned} u \circ \xi(t, r) &= \sqrt{|r - \mu|} \exp \frac{r + t}{2\mu}, \\ v \circ \xi(t, r) &= \sqrt{|r - \mu|} \exp \frac{r - t}{2\mu}, \end{aligned}$$

where (u, v) are the natural coordinates of \mathbb{R}^2 restricted to \mathcal{Q}_K , we find the parametrization of γ as

$$u \circ \xi(t, r) = 1, \quad v \circ \xi(t, r) = -(s + \mu) \exp \frac{-s}{\mu},$$

holding only for $s \in \{s \in \mathbb{R} : -s > \mu\}$. However, because the above equations are solutions of the geodesic differential equation, the uniqueness of the ODE theory ensure that there exists only one analytical extension of these expressions, which is also given⁵ by the latter equations however holding for all $s \in \{s \in \mathbb{R} : -s > 0\}$.

Lastly, because r is defined implicitly in the KS plane by $f(r) = uv$ (cf. Lemma 79),

$$f \circ r \circ \gamma(s) = u \circ \xi(t, r) v \circ \xi(t, r) = -(s + \mu) \exp \frac{-s}{\mu},$$

so that $f \circ r \circ \gamma(0) = -\mu$. Therefore, we conclude that the image of γ always contains points in the black whole, and consequently γ ends in the “fundamental singularity” at “ $r = 0$ ” instead of crossing to \mathcal{R}_{II}^- .

⁴To see why, make a sketch of the “local” lightcones at some isolated points of the Kruskal-Szekeres plane and recall that the tangent vector of timelike geodesics should stays within that lightcones.

⁵The reader may verify it without the ODE theory by using the diffeomorphism $\mathbb{R} \times]0, \mu[\rightarrow \mathcal{R}_I^+$ corresponding to ξ . Recall the proof of Lemma 79.

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